

Question 1

Let $L[y] = y'' + p(t)y' + q(t)y$ on a connected open interval I , where p, q, g are continuous. Suppose y_p is one solution of $L[y] = g$, and u, v form a fundamental set for $L[y] = 0$.

Tasks

1. Prove that every solution of $L[y] = g$ has the form $y_p + Au + Bv$, with unique constants A, B . Prove both directions.
2. If initial data $y(t_0) = a, y'(t_0) = b$ are prescribed, state the initial data that the homogeneous correction must satisfy and explain uniqueness.
3. For $y'' - 3y' + 2y = 2t^2 - 6t + 2$ on $\mathbb{R}$, verify the supplied particular solution $y_p = t^2$ and find the general solution.
4. Solve the example with $y(0) = 1, y'(0) = 0$, verifying the equation and initial data.

Question 1 – Solution

Strategy. Subtract one known forced solution so that the remaining unknown satisfies a homogeneous equation.

Step 1: Prove the representation. If $L[y] = L[y_p] = g$, linearity gives $L[y - y_p] = 0$. Completeness of u, v then gives $y - y_p = Au + Bv$. Conversely, $L[y_p + Au + Bv] = g + A(0) + B(0) = g$. Independence makes the two constants unique for each solution. Thus

$$y = y_p + Au + Bv.$$

Step 2: Transfer the initial data. The correction $h = y - y_p$ must satisfy

$$h(t_0) = a - y_p(t_0), \quad h'(t_0) = b - y_p'(t_0).$$

Regular homogeneous existence and uniqueness determine h on I . Therefore the forced initial-value problem is unique too. The coefficients in the correction must fit these adjusted data, not automatically the original data.

Step 3: Verify and complete the example. For $y_p = t^2$, the left-hand side is $2 - 6t + 2t^2$, as required. The characteristic polynomial of the associated homogeneous equation is $(r - 1)(r - 2)$, so

$$y = t^2 + Ae^t + Be^{2t}.$$

The exponentials are independent homogeneous solutions, and the representation theorem proves that no further solution freedom is missing.

Step 4: Fit and check. At zero, the equations are $A + B = 1$ and $A + 2B = 0$. They give $A = 2$, $B = -1$, hence

$$y = t^2 + 2e^t - e^{2t}.$$

Its initial value is $2 - 1 = 1$ and its slope is $0 + 2 - 2 = 0$. The polynomial contributes the entire forcing, while each exponential has zero residual, verifying the differential equation everywhere.

Question 2

For $y'' + y = 1$ on $\mathbb{R}$, define

$$y_1 = 1, \quad y_2 = 1 + \cos t, \quad y_3 = 1 + \sin t.$$

Consider constant combinations $y = ay_1 + by_2 + cy_3$, where $a, b, c \in \mathbb{R}$.

Tasks

1. Verify all three solutions and find the necessary and sufficient condition on a, b, c for their combination to solve the same forced equation.
2. Prove that every solution of $y'' + y = 1$ has exactly one representation satisfying that condition.
3. Explain whether the solution set is closed under sums, differences and midpoints. Identify the equation satisfied by the difference of two forced solutions.
4. Find the unique coefficients a, b, c producing $y(0) = 0$, $y'(0) = 2$, and verify the resulting solution.

Question 2 – Solution

Strategy. A fixed nonzero forcing is preserved by affine combinations, whose coefficients sum to one.

Step 1: Track the forcing. Each listed function has $y_j'' + y_j = 1$. Therefore

$$L[ay_1 + by_2 + cy_3] = a + b + c.$$

The combination solves the same equation exactly when $a + b + c = 1$.

Step 2: Describe the full set. Subtracting the particular solution 1 leaves $h'' + h = 0$, so all solutions are $1 + B \cos t + C \sin t$. The proposed combination expands to

$$(a + b + c) + b \cos t + c \sin t.$$

It represents the specified solution exactly with $b = B$, $c = C$, $a = 1 - B - C$. Uniqueness follows from the independence of $1, \cos t, \sin t$: if their combination vanishes, adding its second derivative to itself first forces the constant coefficient to zero, then the two harmonic coefficients vanish.

Step 3: Compare operations. The sum of two solutions has forcing 2, and their difference has forcing 0, so neither operation stays in the set with forcing 1. Their midpoint has forcing $(1 + 1)/2 = 1$ and does stay in the set. The zero function is absent. Thus this is an affine translate of a homogeneous solution space, rather than a vector space itself.

Step 4: Fit the data. In $1 + B \cos t + C \sin t$, the data give $B = -1$, $C = 2$. Hence

$$(a, b, c) = (0, -1, 2), \quad y = 1 - \cos t + 2 \sin t.$$

The coefficients sum to one. Directly, $y'' = \cos t - 2 \sin t$, so $y'' + y = 1$; the initial value and slope are 0, 2.

Question 3

For $y'' + 2y' + y = e^{-t}$ on $\mathbb{R}$, two proposed particular solutions are

$$p(t) = \frac{1}{2}t^2e^{-t}, \quad \tilde{p}(t) = e^{-t}(\frac{1}{2}t^2 + 3 - 2t).$$

Two solvers write $y = p + (A + Bt)e^{-t}$ and $y = \tilde{p} + (\alpha + \beta t)e^{-t}$.

Tasks

1. Verify both particular solutions and explain why their difference must be homogeneous.
2. Derive the exact relation between the two pairs of homogeneous constants. Do the two formulas describe different solution sets?
3. Solve $y(0) = 2$, $y'(0) = -1$ in both representations and verify the data.
4. For the selected solution on $t \geq 0$, determine positivity, monotonicity and its limit. Explain why the particular and homogeneous pieces are not uniquely determined by the trajectory.

Question 3 – Solution

Strategy. Changing the particular solution transfers a homogeneous term between the two pieces without changing the full solution.

Step 1: Verify efficiently. For any twice differentiable F , differentiation gives

$$(Fe^{-t})'' + 2(Fe^{-t})' + Fe^{-t} = e^{-t}F''.$$

Both supplied polynomials have second derivative 1, so both are particular solutions. Their difference $(3 - 2t)e^{-t}$ has zero residual, either by this identity or by subtracting the two forced equations.

Step 2: Match coefficients. Equality of the two representations is equivalent to

$$A = \alpha + 3, \quad B = \beta - 2.$$

This is an invertible reassignment of constants, so the two formulas describe the same complete solution family.

Step 3: Impose the data. In the first representation, $y(0) = A$ and $y'(0) = B - A$, because $p(0) = p'(0) = 0$. Thus $A = 2$, $B = 1$, while the second representation has $\alpha = -1$, $\beta = 3$. Both give

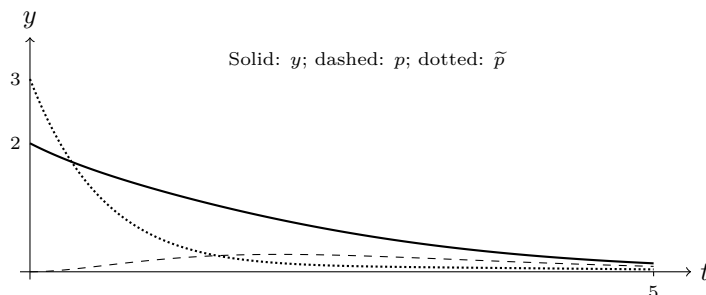
$$y = e^{-t}(2 + t + \tfrac{1}{2}t^2).$$

Its initial value is 2. Differentiation gives $y' = -e^{-t}(1 + \frac{1}{2}t^2)$, whose value at zero is -1 , as required.

Step 4: Interpret the complete trajectory. The derivative follows from subtracting the full polynomial when differentiating its exponential factor:

$$y' = e^{-t}[(1 + t) - (2 + t + \tfrac{1}{2}t^2)] = -e^{-t}(1 + \tfrac{1}{2}t^2).$$

For $t \geq 0$, $y > 0$ and $y' < 0$; the polynomial times e^{-t} tends to zero. The same trajectory admits both decompositions above, so neither piece is unique without a chosen particular solution.



Question 4

For the operator $L[y] = y'' + y$, consider the supplied responses

$$u(t) = \frac{\cos t - \cos 2t}{3}, \quad v(t) = 2(t - \sin t)$$

on $\mathbb{R}$.

Tasks

1. Verify that $L[u] = \cos 2t$, $L[v] = 2t$, and that both responses have zero value and slope at zero.
2. Construct the solution of $y'' + y = \cos 2t + 2t$ with $y(0) = 1$, $y'(0) = -2$. Verify the data.
3. State and prove how both forcing and initial data combine when a constant linear combination of two responses is formed.
4. Let $r = u + \cos t$ and $s = v + \cos t$. Each now has data $(1, 0)$. Correct the naive sum $r + s$ so that it has the combined forcing but only the single prescribed data pair $(1, 0)$.

Question 4 – Solution

Strategy. Superposition combines the initial data as well as the forcing; use a homogeneous correction to achieve the desired data.

Step 1: Verify the supplied responses. We have $u'' = (-\cos t + 4 \cos 2t)/3$, so $u'' + u = \cos 2t$. Also $v' = 2(1 - \cos t)$ and $v'' = 2 \sin t$, so $v'' + v = 2t$. At zero both values and both slopes vanish.

Step 2: Add the forcing and fit the state. The sum $u + v$ handles the combined forcing with zero initial data. Adding the homogeneous solution $\cos t - 2 \sin t$ gives

$$y = u + v + \cos t - 2 \sin t = \frac{4}{3} \cos t - \frac{1}{3} \cos 2t + 2t - 4 \sin t.$$

At zero the value is $4/3 - 1/3 = 1$ and the slope is $2 - 4 = -2$. Linearity verifies the combined forcing, and regular uniqueness proves this is the initial-value solution.

Step 3: Track the full linear map. If $L[y_j] = g_j$ and the initial data are (a_j, b_j) , then $Ay_1 + By_2$ has forcing $Ag_1 + Bg_2$ and data

$$(Aa_1 + Ba_2, Ab_1 + Bb_2).$$

This follows directly from linearity of L , evaluation and differentiation. Ignoring the data part can produce the right equation with the wrong initial state.

Step 4: Remove duplicated initial data. The naive sum $r + s$ has data $(2, 0)$. Subtract one copy of the homogeneous solution $\cos t$:

$$r + s - \cos t = u + v + \cos t.$$

The forcing is unchanged, while the data become $(1, 0)$. This correction differs from Step 2 because the required initial slope here is zero, not -2 .

Question 5

For $y'' - y = 2 \cos t$ on $\mathbb{R}$, a proposed particular solution is $p = -\cos t$. Prescribe real initial data $y(0) = a$, $y'(0) = b$.

Tasks

1. Verify the particular solution and find the full initial-value solution in terms of a, b .
2. Find a necessary and sufficient relation between a, b for boundedness on $[0, \infty)$. For those data, determine the limiting difference $y(t) - p(t)$.
3. Find the unique solution bounded on all of $\mathbb{R}$. Does the selected bounded solution necessarily have a limit as $t \rightarrow \infty$?
4. Compare the trajectories from $(a, b) = (0, -1)$ and $(0, 0)$. Explain why bounded forcing alone does not make every solution bounded.

Question 5 – Solution

Strategy. The difference from a particular solution follows the homogeneous equation, whose growing mode determines boundedness.

Step 1: Verify and fit. Since $p'' = \cos t$, $p'' - p = 2 \cos t$. The general solution is $-\cos t + Ae^t + Be^{-t}$. The data give $A + B = a + 1$, $A - B = b$, hence

$$y = -\cos t + \frac{a+1+b}{2}e^t + \frac{a+1-b}{2}e^{-t}.$$

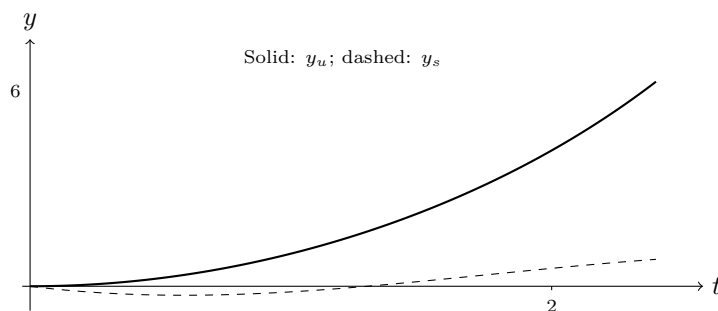
Step 2: Eliminate the growing mode. The bounded oscillatory and decaying terms cannot cancel a nonzero Ae^t as $t \rightarrow \infty$. Thus forward boundedness holds exactly when

$$b = -a - 1.$$

In that case $y = -\cos t + (a+1)e^{-t}$, so $y - p \rightarrow 0$. This convergence is to a time-varying particular solution, not necessarily to a constant.

Step 3: Require both time directions. Boundedness as $t \rightarrow -\infty$ also forces $B = 0$. Thus the unique globally bounded solution is $y = -\cos t$, with data $(-1, 0)$. It has no limit at positive infinity: its values at $2\pi n$ and $(2n+1)\pi$ are -1 and 1 . The forward-bounded family also has these two limiting subsequences because its decaying correction tends to zero.

Step 4: Compare the selected data. Data $(0, -1)$ give $y_s = -\cos t + e^{-t}$, which is bounded forward. Data $(0, 0)$ give $y_u = -\cos t + \cosh t$, which is unbounded forward. Both have the same bounded forcing; the second retains a growing homogeneous component. The finite plot illustrates the distinction; the mode argument proves the global claim.



Question 6

For $y'' + y = \cos t$ on $\mathbb{R}$, a proposed particular solution is $p(t) = \frac{1}{2}t \sin t$. The forcing is bounded in absolute value by one.

Tasks

1. Verify the particular solution and find the full solution family.
2. Solve the initial-value problem with arbitrary data $y(0) = a$, $y'(0) = b$. Identify the zero-data response.
3. Prove that every solution is unbounded above and below on $[0, \infty)$, using two explicit sequences of times. Can a homogeneous correction remove this behavior?
4. For the zero-data response, give exact upper and lower envelope bounds on $t \geq 0$ and the times when they are attained. Explain why envelope contacts need not be stationary points.

Question 6 – Solution

Strategy. Verify the supplied growing response and use exact phase samples to rule out cancellation by bounded homogeneous terms.

Step 1: Verify and complete. Differentiation gives

$$p' = \frac{1}{2} \sin t + \frac{1}{2} t \cos t, \quad p'' = \cos t - \frac{1}{2} t \sin t.$$

Thus $p'' + p = \cos t$ and the general solution is

$$y = A \cos t + B \sin t + \frac{1}{2} t \sin t.$$

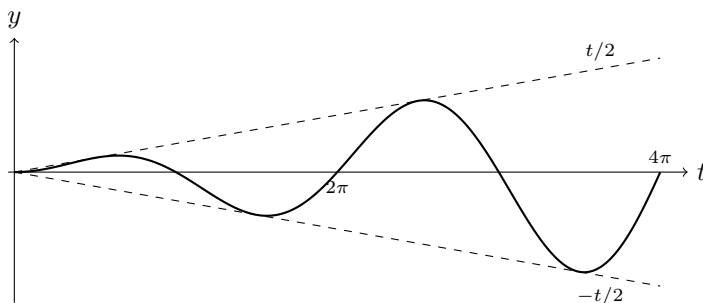
Step 2: Fit the data. Since $p(0) = p'(0) = 0$, the data give $A = a$, $B = b$. In particular the zero-data response is exactly $y_0 = \frac{1}{2} t \sin t$.

Step 3: Prove both unbounded excursions. At $t_n = \pi/2 + 2\pi n$, $n = 0, 1, \dots$, we have $y(t_n) = b + t_n/2 \rightarrow \infty$. At $s_n = 3\pi/2 + 2\pi n$, we have $y(s_n) = -b - s_n/2 \rightarrow -\infty$. These values do not depend on a . Thus no constant homogeneous correction removes the unbounded behavior, even though the forcing is bounded.

Step 4: Distinguish bounds from extrema. On $t \geq 0$,

$$-t/2 \leq y_0(t) \leq t/2.$$

For $t > 0$, the upper envelope is attained exactly at t_n and the lower at s_n ; both bounds also coincide with y_0 at zero. At a positive upper contact $y'_0 = 1/2$, and at a lower contact $y'_0 = -1/2$. These slopes match those of the respective envelopes and are not zero, so the contacts are not stationary points of y_0 .



Question 7

On $t > 0$, consider

$$y'' = 1/t, \quad y(1) = 0, \quad y'(1) = 0,$$

with supplied particular solution $p(t) = t \ln t - t$. The associated homogeneous equation has the globally smooth solutions $1, t$.

Tasks

1. Verify the particular solution and find every solution on $(0, \infty)$.
2. Solve the initial-value problem and determine its monotonicity, convexity and minimum.
3. Find the right-hand limit at zero and decide whether the selected solution admits continuous, C^1 , or C^2 extension through zero.
4. Can any homogeneous correction to p produce a C^1 extension through zero? Explain why smooth homogeneous solutions do not guarantee continuation of the forced problem.

Question 7 – Solution

Strategy. The forcing can impose endpoint behavior that no homogeneous correction can cancel.

Step 1: Verify and complete. Since $p' = \ln t$ and $p'' = 1/t$, it is a particular solution. Adding all affine homogeneous functions gives

$$y = t \ln t - t + At + B \quad (t > 0).$$

Step 2: Fit and analyze. The slope at 1 gives $A = 0$, and the value then gives $B = 1$. Thus

$$y = t \ln t - t + 1, \quad y' = \ln t, \quad y'' = 1/t > 0.$$

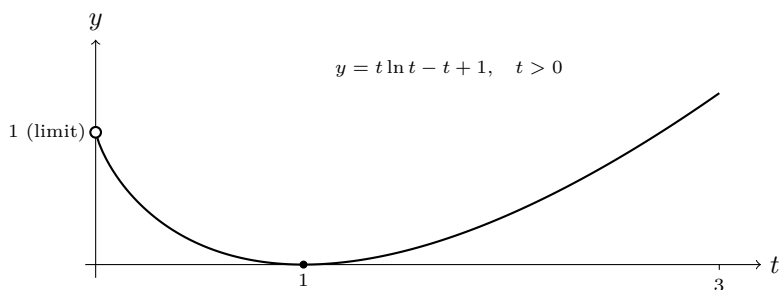
The function decreases on $(0, 1)$, increases on $(1, \infty)$, and is strictly convex. Its unique global minimum is $y(1) = 0$.

Step 3: Test extension of the selected function. Since $t \ln t \rightarrow 0$ as $t \downarrow 0$, we have $y(t) \rightarrow 1$. Setting $y(0) = 1$ and, for example, $y(t) = 1$ for $t < 0$ gives a continuous extension. A finite derivative at zero is impossible because

$$\frac{y(t) - 1}{t} = \ln t - 1 \rightarrow -\infty.$$

Hence there is no C^1 extension and therefore no C^2 extension. The limiting point in the graph is excluded from the original domain.

Step 4: Test every correction. For general A, B , continuity would require $y(0) = B$, but the difference quotient is $\ln t - 1 + A$, again tending to $-\infty$. No finite homogeneous constants cure this. Although $1, t$ extend smoothly, the forcing $1/t$ is undefined at zero. The maximal regular interval containing the initial point is $(0, \infty)$, not all of $\mathbb{R}$.



Question 8

Let u, v satisfy the same forced equation $y'' + p(t)y' + q(t)y = g(t)$ on a regular interval, and write $W = uv' - u'v$. For a concrete example on $\mathbb{R}$, take

$$u = 1 - \cos t, \quad v = 1 - \cos t + \sin t.$$

Tasks

1. Derive the differential identity for W when the common forcing is nonzero.
2. Verify that the example functions solve $y'' + y = 1$. Compute their Wronskian and explain why its zero at zero does not force it to vanish everywhere.
3. Prove that the two example functions are linearly independent as functions, and identify the homogeneous solution obtained from their difference.
4. Classify the constant affine combinations of u, v that solve the same forced equation. Which initial values and slopes at zero can they attain, and do they give the entire forced solution set?

Question 8 – Solution

Strategy. Recompute the Wronskian identity with the forcing included, then distinguish linear independence from a homogeneous fundamental set.

Step 1: Keep the forcing terms. Differentiating and substituting gives

$$W' = uv'' - u''v = u(g - pv' - qv) - v(g - pu' - qu),$$

so

$$W' = -pW + g(u - v).$$

The extra term is absent from Abel's homogeneous identity and generally does not vanish.

Step 2: Verify the example. We have $u'' + u = \cos t + 1 - \cos t = 1$, and adding $\sin t$ preserves this forcing. Direct computation gives

$$W = (1 - \cos t) \cos t - \sin^2 t = \cos t - 1.$$

It is zero at 0 but equals -2 at π . Its derivative is $-\sin t$, matching $g(u - v) = -\sin t$. The homogeneous zero-or-never-zero conclusion is inapplicable to this forced pair.

Step 3: Test independence. If $Au + Bv = 0$, evaluation at π gives $2A + 2B = 0$ and at $\pi/2$ gives $A + 2B = 0$. Hence $A = B = 0$. The difference $v - u = \sin t$ solves the associated homogeneous equation. Independence of the forced functions does not make them a fundamental set of homogeneous solutions.

Step 4: Describe their affine family. The forcing of $Au + Bv$ is $A + B$, so retaining forcing 1 requires $A + B = 1$. Such combinations are

$$u + B \sin t = 1 - \cos t + B \sin t.$$

They have initial value zero and arbitrary slope B . They give exactly the forced solutions with initial value zero, but miss every solution with a different initial value. The full family is $1 + C \cos t + D \sin t$, with two independent initial-data freedoms.

Question 9

Let y solve $y'' + y = 1$ with $y(0) = y'(0) = 0$. On $0 \leq t \leq 1$, consider the approximation $Y(t) = t^2/2$ and the error $E = Y - y$. You may use $0 \leq \sin r \leq r$ for $0 \leq r \leq 1$.

Tasks

1. Compute the residual $R = Y'' + Y - 1$ and derive the differential equation and initial data satisfied by E .
2. Verify by differentiating twice that $\int_0^t \sin(t-s)s^2/2 \, ds$ solves this error initial-value problem. Justify that it equals E .
3. Prove the rigorous error bound $0 \leq E(t) \leq t^4/24$ on $[0, 1]$, including the direction of the approximation error.
4. Find the exact solution y by verification, compute the actual error at 1 to six decimal places, and compare it with the bound. Why does a residual vanishing only at zero not prove exactness?

Question 9 – Solution

Strategy. Treat the approximation residual as a forcing for the error, then bound the resulting integral with a positive kernel.

Step 1: Form the error equation. Since $Y'' = 1$, the residual is $R = t^2/2$. Subtraction gives

$$E'' + E = t^2/2, \quad E(0) = E'(0) = 0.$$

Step 2: Verify the integral. Set $F(t) = \int_0^t \sin(t-s)s^2/2 \, ds$. Leibniz differentiation gives

$$F' = \int_0^t \cos(t-s)s^2/2 \, ds, \quad F'' = t^2/2 - \int_0^t \sin(t-s)s^2/2 \, ds.$$

Thus $F'' + F = t^2/2$ and $F(0) = F'(0) = 0$. The continuous coefficients and forcing give uniqueness, so $F = E$.

Step 3: Bound the error. For $0 \leq s \leq t \leq 1$, the supplied sine inequality gives

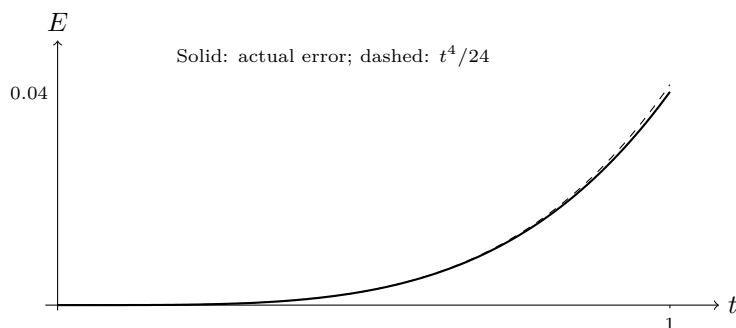
$$0 \leq E(t) \leq \frac{1}{2} \int_0^t (t-s)s^2 \, ds = \frac{1}{2}(t^4/3 - t^4/4) = \boxed{t^4/24}.$$

Hence Y overestimates the solution. The bound vanishes at zero and remains valid across the entire stated interval.

Step 4: Check against the exact response. The function $y = 1 - \cos t$ has the required equation and zero data. Therefore

$$E(1) = \cos 1 - \frac{1}{2} \approx 0.040302 < 1/24 \approx 0.041667.$$

The residual is zero at the initial point but is positive elsewhere on $(0, 1]$. A solution must satisfy its equation at every point, so agreement at one point and in the initial data does not make Y exact.



Question 10

On $0 \leq t \leq 1$, design a forcing g for $y'' + 2y' + y = g(t)$ so that the trajectory has

$$y(0) = 0, \quad y'(0) = 0, \quad y(1) = 1, \quad y'(1) = 0.$$

Restrict the desired trajectory to polynomials of degree at most three.

Tasks

1. Find the unique polynomial meeting the four endpoint requirements.
2. Derive the forcing g that makes this polynomial an exact solution and verify all four requirements.
3. Prove that this forcing changes sign exactly once on $[0, 1]$, locating the zero between $3/4$ and $4/5$. Can a nonnegative forcing realize this same cubic trajectory?
4. Keep the designed forcing but perturb the initial data to $y(0) = \varepsilon$, $y'(0) = 0$. Find the exact position and slope errors at $t = 1$, and determine whether the target state is still reached when $\varepsilon \neq 0$.

Question 10 – Solution

Strategy. Prescribe a trajectory and compute its forcing directly; then use a homogeneous difference equation to analyze initial-data error.

Step 1: Fit the cubic trajectory. Write $y = c_0 + c_1t + c_2t^2 + c_3t^3$. The initial conditions give $c_0 = c_1 = 0$. The final conditions give $c_2 + c_3 = 1$ and $2c_2 + 3c_3 = 0$, so

$$y_*(t) = 3t^2 - 2t^3.$$

The coefficient equations have a unique solution, including among all polynomials of lower degree.

Step 2: Compute the forcing. The derivatives are $y'_* = 6t - 6t^2$ and $y''_* = 6 - 12t$. Thus

$$g = y''_* + 2y'_* + y_* = 6 - 9t^2 - 2t^3.$$

The derivative formulas give zero initial value and slope, and final value $3 - 2 = 1$ with slope $6 - 6 = 0$. The computed forcing verifies the equation identically.

Step 3: Locate the sign change. We have $g(0) = 6$, $g(1) = -5$, and $g' = -18t - 6t^2 < 0$ for $t > 0$. Therefore there is exactly one zero. More precisely,

$$g(3/4) = 3/32 > 0, \quad g(4/5) = -98/125 < 0,$$

so the zero lies strictly between $3/4$ and $4/5$. This same cubic fixes g pointwise, so a nonnegative forcing cannot realize it on the whole interval. This conclusion concerns the specified cubic, not every possible trajectory.

Step 4: Propagate the initial error. The difference $h = y - y_*$ satisfies $h'' + 2h' + h = 0$, with $h(0) = \varepsilon$, $h'(0) = 0$. Hence $h = \varepsilon(1 + t)e^{-t}$ and $h' = -\varepsilon te^{-t}$. At the endpoint,

$$y(1) - 1 = 2\varepsilon/e, \quad y'(1) = -\varepsilon/e.$$

For nonzero ε the target state is missed. The forcing is unchanged; the error evolves entirely through the homogeneous correction.