

Question 1

For two real solutions u, v of $y'' + p(t)y' + q(t)y = 0$ on a connected open interval I , define the ordered Wronskian

$$W[u, v](t) = u(t)v'(t) - u'(t)v(t).$$

Assume p, q are continuous and $t_0 \in I$.

Tasks

1. Derive a first-order equation for W and solve it in terms of $W(t_0)$. Your derivation must include the case $W(t_0) = 0$.
2. Prove that W either vanishes everywhere on I or never vanishes there. Explain the consequences for independence of the two solutions.
3. For $(1 + t^2)y'' + 2ty' + (1 + t^2)y = 0$ on $\mathbb{R}$, find $W(t)$ if $W(0) = -3$, without solving the second-order equation.
4. Find the sign and limiting values of this Wronskian. Does its approach to zero at infinity imply that the solutions become dependent at a finite time?

Question 1 – Solution

Strategy. Differentiate the determinant and substitute the common equation before integrating.

Step 1: Derive Abel's identity. Differentiation cancels the mixed terms:

$$W' = uv'' - u''v = u(-pv' - qv) - (-pu' - qu)v = -pW.$$

Multiplication by $\exp(\int_{t_0}^t p(s) ds)$ gives a function with zero derivative. Thus, without dividing by W ,

$$W(t) = W(t_0) \exp\left[-\int_{t_0}^t p(s) ds\right].$$

This includes $W \equiv 0$ when $W(t_0) = 0$.

Step 2: Use the nonzero exponential. The exponential factor is positive and finite at every point of I . A nonzero initial Wronskian stays nonzero with the same sign. If it is zero, the initial-data columns are dependent; a nontrivial constant combination has zero value and slope at t_0 . Regular uniqueness forces that combination to vanish on I . Therefore, for solutions of this common regular equation, $W \equiv 0$ is equivalent to dependence.

Step 3: Normalize the example. Divide by $1 + t^2$, giving $p = 2t/(1 + t^2)$. Integration gives

$$\int_0^t \frac{2s}{1+s^2} ds = \ln(1+t^2), \quad W(t) = -\frac{3}{1+t^2}.$$

Thus the Wronskian is determined without finding either solution. The undivided coefficient $2t$ is not the coefficient to use in Abel's identity.

Step 4: Interpret the limits. We have $W < 0$ everywhere, with $W \rightarrow 0$ from below as $t \rightarrow \pm\infty$. The pair is independent on all of $\mathbb{R}$, and its data columns are independent at every finite time. A limiting zero outside the interval is not a zero at a point where the initial-data criterion applies.

Question 2

Let u, v solve $y'' + 2y' + q(t)y = 0$ on $\mathbb{R}$, where q is continuous, with data

$$(u(0), u'(0)) = (1, 0), \quad (v(0), v'(0)) = (0, 1).$$

Define $f = 2u - v$ and $g = 3u + 4v$. At each time, regard (f, f') and (g, g') as vectors in the value-slope plane.

Tasks

1. For arbitrary constants a, b, c, d , derive the formula relating $W[au + bv, cu + dv]$ to $W[u, v]$.
2. Find $W[f, g](t)$ and decide whether f, g form a fundamental set. Explain the effect of reversing their order.
3. Interpret the Wronskian as oriented parallelogram area. Find the ordinary area at time zero and the first nonnegative time at which it is half that size.
4. Can the two state vectors become parallel at a finite time? Explain what their area tending to zero does and does not imply.

Question 2 – Solution

Strategy. Separate the constant determinant of a basis change from the time evolution supplied by Abel's identity.

Step 1: Expand the determinant. Terms containing uu' or vv' cancel, leaving

$$W[au + bv, cu + dv] = (ad - bc)W[u, v].$$

The constants may be factored through differentiation; this step would require extra terms for time-dependent coefficients.

Step 2: Apply the two determinants. Here $W[u, v](0) = 1$ and $p = 2$, so $W[u, v] = e^{-2t}$. The coefficient determinant is $2 \cdot 4 - (-1) \cdot 3 = 11$, hence

$$W[f, g] = 11e^{-2t} > 0.$$

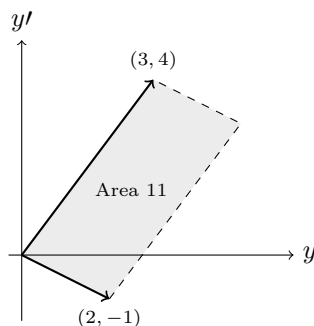
Both combinations solve the equation and are independent, so they are fundamental. Reversing their order negates the Wronskian but leaves independence unchanged.

Step 3: Compute the area. The columns (f, f') , (g, g') have signed area $W[f, g]$ and ordinary area $|W[f, g]|$. At zero they are $(2, -1)$ and $(3, 4)$, with area 11. The half-area equation gives

$$11e^{-2t} = 11/2, \quad t = \frac{1}{2} \ln 2.$$

The area is strictly decreasing for $t \geq 0$, so this is the first such time.

Step 4: Distinguish collapse from a limit. The determinant never vanishes at finite t , so the two state vectors never become parallel there. The area tends to zero as $t \rightarrow \infty$; this does not destroy finite-time independence. Area alone also does not specify the lengths of the two vectors or their angle separately.



Question 3

Consider the functions $u(t) = t^3$ and $v(t) = |t|^3$ on $\mathbb{R}$. A student claims that a Wronskian which is identically zero always proves linear dependence of two C^2 functions.

Tasks

1. Prove that both functions are C^2 on $\mathbb{R}$, explicitly checking the derivatives of v at zero.
2. Compute $W[u, v]$ everywhere and determine whether the two functions are linearly independent on $\mathbb{R}$.
3. Explain precisely why the common regular-equation theorem does not contradict your result. Could both functions solve a normalized equation with continuous coefficients on $\mathbb{R}$?
4. Verify that both solve the undivided equation $t^2y'' - 2ty' = 0$ on $\mathbb{R}$. Compare the ratio v/u on the two sides of zero.

Question 3 – Solution

Strategy. Check the counterexample directly, then identify the hypothesis missing from the student's assertion.

Step 1: Check regularity. For $t \neq 0$, $v' = 3t|t|$ and $v'' = 6|t|$. At zero,

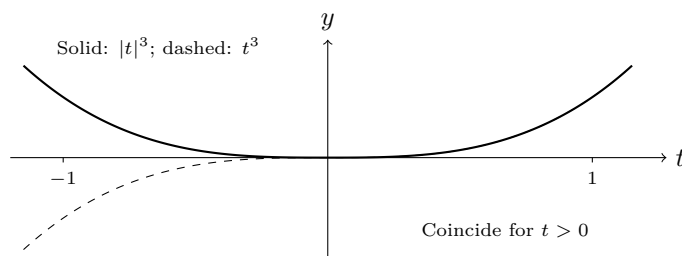
$$v'(0) = \lim_{h \rightarrow 0} |h|^3/h = 0, \quad v''(0) = \lim_{h \rightarrow 0} 3h|h|/h = 0.$$

These agree continuously with the formulas away from zero, so v is C^2 ; u is a polynomial and is smooth.

Step 2: Compare the Wronskian and independence. On $t > 0$, $v = u$; on $t < 0$, $v = -u$. Thus $W = 0$ on each side, and at zero both data columns are zero, so $\boxed{W \equiv 0}$. Yet $Au + Bv = 0$ on all of $\mathbb{R}$ forces $A + B = 0$ on the positive side and $A - B = 0$ on the negative side. Hence $A = B = 0$: the functions are independent on $\mathbb{R}$.

Step 3: Restore the missing hypothesis. The zero-Wronskian implication requires solutions of one common regular homogeneous linear equation on a connected interval, not merely two C^2 functions. The nonzero solution u would have zero value and slope at zero, contradicting regular uniqueness. Thus no normalized equation with continuous coefficients on $\mathbb{R}$ can have both as solutions.

Step 4: Verify a singular equation. For u , the residual is $t^2(6t) - 2t(3t^2) = 0$. On each side $v = \pm u$, so its residual also vanishes; at zero both terms vanish. The leading coefficient is zero there. The ratio v/u is 1 on $t > 0$ and -1 on $t < 0$, and is undefined at the shared zero. Local constant ratios need not join into one global constant.



Question 4

A regular equation $y'' + p(t)y' + q(t)y = 0$ on $\mathbb{R}$ has a known solution $u = e^t$. A desired companion v must satisfy

$$W[u, v](t) = 2e^{-t^2}, \quad v(0) = 0.$$

Assume p, q are continuous; an unevaluated definite integral is an acceptable exact answer.

Tasks

1. Determine the only possible coefficients p, q .
2. Construct v using the prescribed Wronskian and initial value, and find $v'(0)$.
3. Verify both the Wronskian and the differential equation without evaluating the integral.
4. If the known solution u were not supplied, would the prescribed Wronskian alone determine q ? Justify your answer using initial-data normalization for arbitrary continuous q .

Question 4 – Solution

Strategy. Abel's identity determines the first-derivative coefficient; the known nonzero solution then determines the remaining coefficient.

Step 1: Recover the equation. Since W never vanishes,

$$p = -W'/W = 2t.$$

Substituting $u = e^t$ gives $1 + p + q = 0$, hence

$$y'' + 2ty' - (1 + 2t)y = 0.$$

Step 2: Recover the companion. The identity $W = e^t(v' - v)$ gives $v' - v = 2e^{-t^2-t}$. Multiplying by e^{-t} and integrating from zero yields

$$v(t) = e^t \int_0^t 2e^{-s^2-2s} ds.$$

The integral is smooth on all of $\mathbb{R}$. It gives $v(0) = 0$ and $v'(0) = 2$.

Step 3: Verify by differentiation. Write J for the integral, so $J' = 2e^{-t^2-2t}$. Then $v' = v + 2e^{-t^2-t}$ and $e^t(v' - v) = 2e^{-t^2}$, as prescribed. If $L[y] = y'' + py' + qy$, direct expansion gives

$$uL[v] - vL[u] = W' + pW.$$

Here $L[u] = 0$ and $W' + 2tW = 0$. Since $u = e^t \neq 0$, this proves $L[v] = 0$ everywhere.

Step 4: Identify what the Wronskian omits. With $p = 2t$ and any continuous choice of q , regular existence supplies solutions with data $(1, 0)$ and $(0, 2)$ at zero. Their initial Wronskian is 2, so Abel's identity gives the same $2e^{-t^2}$ for every such q . Thus the Wronskian alone fixes p but not q ; the supplied nonzero seed was essential for the latter.

Question 5

Two independent solutions of a regular equation $y'' + p(t)y' + q(t)y = 0$ on $\mathbb{R}$ have measured ordered Wronskians

$$W(0) = 6, \quad W(2) = 3, \quad W(5) = 12.$$

Assume p, q are continuous and the measurements are exact.

Tasks

1. Find $\int_0^2 p(t) dt$ and $\int_2^5 p(t) dt$, and the average value of p on each interval.
2. Prove that p is positive somewhere in $(0, 2)$, negative somewhere in $(2, 5)$, and zero somewhere in $(0, 5)$.
3. Under the additional assumption that $p(t) = a + bt$, determine a, b and the full Wronskian function. Check all three measurements.
4. For this affine model, find the unique minimum of W on $[0, 5]$. Explain why three Wronskian samples alone do not prove that the true coefficient is affine.

Question 5 – Solution

Strategy. Ratios of Wronskian measurements give integrals of the coefficient, rather than its pointwise values.

Step 1: Extract the averages. Abel's identity gives

$$\int_0^2 p = \ln 2, \quad \int_2^5 p = -\ln 4.$$

The average values are $\boxed{(\ln 2)/2}$ and $\boxed{-(\ln 4)/3}$ respectively.

Step 2: Force a sign change. A continuous function with positive integral must be positive at an interior point; likewise the negative integral requires an interior negative value. Choosing one point from each interval, the intermediate value theorem supplies a zero between them. This conclusion does not specify the value of p at time 2.

Step 3: Fit the affine model. The coefficient equations are $2a + 2b = \ln 2$ and $3a + (21/2)b = -2\ln 2$. Solving gives

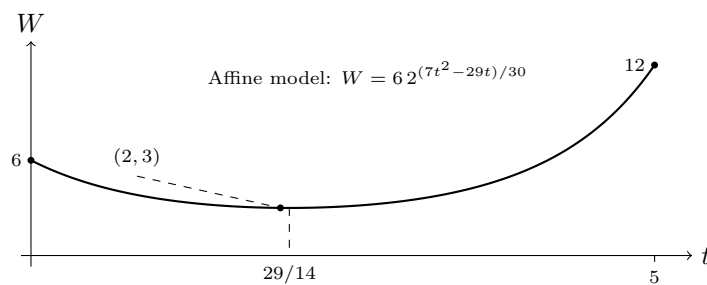
$$p(t) = \frac{29 - 14t}{30} \ln 2, \quad \boxed{W(t) = 6 \cdot 2^{(7t^2 - 29t)/30}}.$$

At $t = 0, 2, 5$, the exponents are 0, -1 , 1, giving 6, 3, 12 as required.

Step 4: Locate and qualify the minimum. Since $W' = -pW$ and $W > 0$, the Wronskian decreases until $t = 29/14$ and increases afterward. Its unique minimum on $[0, 5]$ is

$$\boxed{W(29/14) = 6 \cdot 2^{-841/840}}.$$

Affineness is an extra modeling assumption. For example, adding any multiple of $\cos(2\pi t)$ to this p leaves both measured integrals unchanged, but generally gives a nonaffine coefficient and a different Wronskian between the measurement times.



Question 6

Let u, v be independent real solutions of $y'' + p(t)y' + q(t)y = 0$ on an open interval I , with continuous coefficients. Suppose $a < b$ lie in I and are consecutive zeros of u : $u(a) = u(b) = 0$ and $u(t) \neq 0$ for $a < t < b$.

Tasks

1. Prove that the zeros at a, b are simple and that $v(a), v(b)$ are nonzero.
2. Use the sign of the Wronskian at the endpoints to prove that $v(a)$ and $v(b)$ have opposite signs.
3. Differentiate v/u on (a, b) and use it to prove that v has exactly one zero there.
4. Apply the result to $u = \sin t$, $v = \cos t + 2 \sin t$ between 0 and π , verifying independence and finding the unique zero explicitly.

Question 6 – Solution

Strategy. Use the constant sign of the Wronskian first at the endpoints and then in the derivative of a ratio.

Step 1: Exclude repeated and shared zeros. If u and u' both vanished at an endpoint, regular uniqueness would make u identically zero, contrary to independence. Thus both zeros are simple. Since W never vanishes, $W(a) = -u'(a)v(a)$ and $W(b) = -u'(b)v(b)$ force $v(a), v(b) \neq 0$.

Step 2: Compare endpoint signs. Multiply u by -1 if necessary so that $u > 0$ on (a, b) . Simplicity then implies $u'(a) > 0$ and $u'(b) < 0$. Abel's identity says $W(a), W(b)$ have the same sign. The two endpoint formulas therefore force $v(a), v(b)$ to have opposite signs. Continuity gives at least one zero of v inside.

Step 3: Prove uniqueness of that zero. On the zero-free interval for u ,

$$(v/u)' = \frac{uv' - u'v}{u^2} = \frac{W}{u^2}.$$

The derivative has a fixed nonzero sign, so the ratio is strictly monotone. Its zeros are precisely the zeros of v there. It can vanish at most once; combined with the preceding existence argument, this proves exactly one zero.

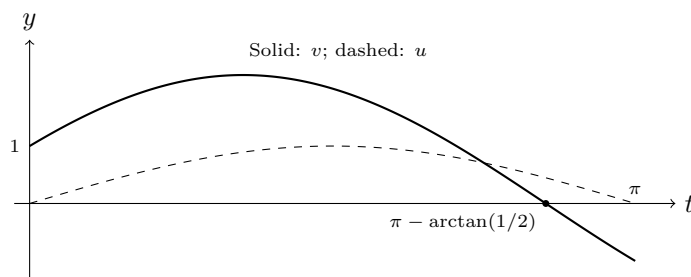
Step 4: Compute the example. Both candidates solve $y'' + y = 0$, and

$$W[\sin t, \cos t + 2 \sin t] = -1.$$

The endpoint values of v are $1, -1$. In $(0, \pi)$, the equation $\cos t + 2 \sin t = 0$ has its root in the second quadrant, with

$$t = \pi - \arctan(1/2).$$

The separation argument proves there are no other roots between these consecutive zeros of $\sin t$.



Question 7

Let p be C^1 and q continuous on an open interval I containing zero, and set $P(t) = \int_0^t p(s) ds$. Consider the substitution $y = e^{-P/2}z$ in $y'' + py' + qy = 0$.

Tasks

1. Derive the differential equation satisfied by z and show that its first-derivative term vanishes.
2. For two transformed solutions, derive the relation between their Wronskian and the original pair's Wronskian. Explain why the transformed Wronskian is constant.
3. Apply the method to $y'' + 2ty' + (t^2 + 3)y = 0$ on $\mathbb{R}$. Construct a fundamental pair normalized to data $(1, 0)$ and $(0, 1)$ at zero and compute its Wronskian.
4. Solve $y(0) = 1$, $y'(0) = 2$ and explain why both members of your fundamental pair tending to zero as $t \rightarrow \infty$ does not contradict independence.

Question 7 – Solution

Strategy. A common nonzero multiplier changes the Wronskian by its square and can remove the first-derivative coefficient.

Step 1: Differentiate the substitution. Put $h = e^{-P/2}$, so $h'/h = -p/2$ and $h''/h = -p'/2 + p^2/4$. Substitution of $y = hz$ gives

$$z'' + (2h'/h + p)z' + (h''/h + ph'/h + q)z = 0,$$

and therefore

$$z'' + [q - p'/2 - p^2/4]z = 0.$$

Step 2: Transform the determinant. Expanding derivatives cancels the terms containing hh' :

$$W[hz_1, hz_2] = h^2 W[z_1, z_2] = e^{-P} W[z_1, z_2].$$

The transformed equation has no first-derivative term, so its Wronskian has derivative zero. The nonzero multiplier preserves independence.

Step 3: Solve the transformed example. Here $P = t^2$ and $q - p'/2 - p^2/4 = t^2 + 3 - 1 - t^2 = 2$. Thus $z'' + 2z = 0$, and the normalized pair is

$$\phi = e^{-t^2/2} \cos(\sqrt{2}t), \quad \psi = e^{-t^2/2} \frac{\sin(\sqrt{2}t)}{\sqrt{2}}.$$

At zero, $h = 1$ and $h' = 0$, so the data remain $(1, 0)$ and $(0, 1)$. The transformed Wronskian is 1, giving $W[\phi, \psi] = e^{-t^2} > 0$ everywhere.

Step 4: Match and interpret. The normalized data give

$$y = \phi + 2\psi = e^{-t^2/2} [\cos(\sqrt{2}t) + \sqrt{2} \sin(\sqrt{2}t)].$$

Its value and slope at zero are 1, 2. Both basis functions decay to zero because their trigonometric factors are bounded and the Gaussian factor decays. Dependence would require an exact constant relation on the interval; the strictly positive finite-time Wronskian rules that out.

Question 8

Suppose p, q are real continuous functions on $\mathbb{R}$, each periodic with period $T > 0$. Consider $y'' + p(t)y' + q(t)y = 0$. A T -periodic solution satisfies $y(t + T) = y(t)$ for every real t .

Tasks

1. Prove that the existence of two independent T -periodic solutions requires $\int_0^T p(t) dt = 0$.
2. If $p \geq 0$ and p is not identically zero, decide whether two such periodic solutions can exist.
3. Is the zero-integral condition sufficient? Analyze $y'' - y = 0$ as a counterexample and justify the absence of nonzero periodic solutions.
4. Does the existence of just one nonzero T -periodic solution force the same integral condition? Analyze $y'' + y' = 0$ and identify a nonperiodic companion.

Question 8 – Solution

Strategy. Periodicity forces the Wronskian to return to its starting value, while Abel's identity specifies its multiplicative change.

Step 1: Compare one full period. Differentiating $y(t+T) = y(t)$ shows that the slope is periodic too. Thus two periodic solutions have $W(T) = W(0)$. Independence gives $W(0) \neq 0$, so Abel's identity implies

$$\exp\left[-\int_0^T p(t) dt\right] = 1.$$

The integral is real; consequently

$$\int_0^T p(t) dt = 0.$$

Step 2: Exclude nontrivial nonnegative damping. By continuity and periodicity, if $p \geq 0$ and is not identically zero, it is positive on some interval within a period. Hence its integral over a period is strictly positive. The necessary condition fails, so two independent T -periodic solutions cannot exist.

Step 3: Disprove sufficiency. For $y'' - y = 0$, $p = 0$ satisfies the integral condition for every T . Every solution is $Ae^t + Be^{-t}$. A continuous periodic function is bounded on $\mathbb{R}$; boundedness as $t \rightarrow \infty$ forces $A = 0$, and boundedness as $t \rightarrow -\infty$ forces $B = 0$. Thus there are no nonzero periodic solutions, much less two independent ones.

Step 4: Test a single periodic solution. In $y'' + y' = 0$, the constant solution $u = 1$ is nonzero and periodic for every $T > 0$, while $p = 1$ has integral $T \neq 0$. A companion is $v = e^{-t}$, with $W[u, v] = -e^{-t} \neq 0$; it is not periodic. One periodic solution does not make a nonzero pair Wronskian periodic and imposes no such zero-integral requirement.

Question 9

On $\mathbb{R}$, consider the original undivided equation

$$t^2 y'' - 2ty' + 2y = 0$$

and the smooth candidates $u = t, v = t^2$. A classical global solution here means a C^2 function on $\mathbb{R}$ satisfying this undivided equation at every point.

Tasks

1. Verify both candidates and compute their Wronskian. Determine their independence on $\mathbb{R}$ and identify every zero of the Wronskian.
2. Apply Abel's identity separately on the two largest regular intervals. Explain why its zero-or-never-zero conclusion cannot be applied across zero.
3. Find every classical global solution by matching the general solutions from the negative and positive sides through zero.
4. Find all global solutions with $y(0) = 0, y'(0) = 1$. Explain both the failure of uniqueness and why no regular normalized equation through zero can have this candidate pair as solutions.

Question 9 – Solution

Strategy. The singular leading coefficient changes which interval hypotheses are available; explicitly match derivatives to determine global solutions.

Step 1: Verify and compute. The residuals are $-2t + 2t = 0$ and $2t^2 - 4t^2 + 2t^2 = 0$. Also

$$W[t, t^2] = t(2t) - 1(t^2) = t^2.$$

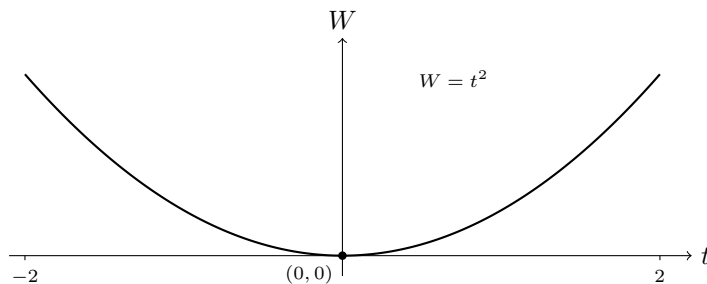
It is zero only at $t = 0$. The functions are independent on $\mathbb{R}$, since $At + Bt^2 \equiv 0$ forces $A = B = 0$.

Step 2: Respect the regular intervals. Division by t^2 gives $p = -2/t$, $q = 2/t^2$, continuous on $(-\infty, 0)$ and $(0, \infty)$. On either interval Abel's identity gives $W = Ct^2$, with the constant fixed within that interval. For this pair it is $C = 1$ on both sides. There is no regular interval containing zero. Abel's conclusion applies separately on the two intervals and makes no assertion across zero.

Step 3: Match all global solutions. The pair is fundamental on either side, so write $A_-t + B_-t^2$ for $t < 0$ and $A_+t + B_+t^2$ for $t > 0$. Continuity forces $y(0) = 0$. Matching first derivatives forces $A_- = A_+$; matching second derivatives forces $B_- = B_+$. Conversely every resulting polynomial solves the equation globally. Thus

$$y = At + Bt^2 \quad \text{on } \mathbb{R}.$$

Step 4: Inspect the singular data. The data give $A = 1$ but leave B arbitrary, so all solutions are $y = t + Bt^2$. Uniqueness fails despite this two-dimensional global family. No regular normalized equation through zero could include $v = t^2$: at zero its value and slope vanish while its second derivative is 2, making the normalized equation impossible there.



Question 10

Let u, v be C^2 real functions on an interval and let θ be a real C^2 function. Define time-dependent combinations

$$f = u \cos \theta + v \sin \theta, \quad g = -u \sin \theta + v \cos \theta.$$

At every time the two-by-two coefficient matrix has determinant 1.

Tasks

1. Derive the actual expression for $W[f, g]$ in terms of u, v, θ and their derivatives. Explain why the constant change-of-basis formula cannot simply be used.
2. Take $u = \cos t$, $v = \sin t$, $\theta = t$ on $\mathbb{R}$. Compute f, g and their Wronskian. Does the pointwise invertibility preserve a fundamental set for $y'' + y = 0$?
3. Keep the same u, v but take $\theta = t/2$. Show that a nonzero resulting Wronskian still does not make the new pair solutions of the original equation.
4. For these trigonometric u, v , classify every real C^2 function θ on $\mathbb{R}$ for which both f, g solve $y'' + y = 0$.

Question 10 – Solution

Strategy. Differentiate the moving coefficients too; pointwise invertibility is not a constant linear transformation of solution functions.

Step 1: Include the extra derivative terms. In the derivatives of f, g , the terms involving u', v' contribute $W[u, v]$, as in a constant rotation. The remaining terms are $\theta'g$ in f' and $-\theta'f$ in g' . Hence

$$W[f, g] = W[u, v] - \theta'(f^2 + g^2) = W[u, v] - \theta'(u^2 + v^2).$$

The additional term is exactly what the constant-coefficient formula omits.

Step 2: Observe a complete collapse. For $u = \cos t, v = \sin t$, the addition formulas give $f = \cos(t - \theta)$, $g = \sin(t - \theta)$. At $\theta = t$, $f = 1$ and $g = 0$, so $W = 0$. They are dependent, and $f'' + f = 1 \neq 0$. Pointwise invertibility of the moving coefficient matrix does not preserve the solution space.

Step 3: Test a nonzero determinant. At $\theta = t/2$, the functions are $\cos(t/2), \sin(t/2)$ and $W = 1/2$. Each has $y'' + y = (3/4)y$, so neither solves the original equation identically. They instead form a fundamental set for $y'' + y/4 = 0$.

Step 4: Classify all valid rotations. Put $\varphi = t - \theta$. The residuals for $f = \cos \varphi, g = \sin \varphi$ are

$$(1 - \varphi'^2) \cos \varphi - \varphi'' \sin \varphi, \quad (1 - \varphi'^2) \sin \varphi + \varphi'' \cos \varphi.$$

Both vanish exactly when $\varphi'^2 = 1$ and $\varphi'' = 0$, since the displayed rotation is invertible. On the connected real line this gives $\varphi = \pm t + C$, hence

$$\theta(t) = C \quad \text{or} \quad \theta(t) = 2t + C, \quad C \in \mathbb{R}.$$

In the second case the final functions are still constant combinations of $\cos t, \sin t$, despite their time-dependent defining coefficients.