

Question 1

A fundamental set for a homogeneous linear equation on an interval is a pair of solutions whose constant linear combinations represent every solution uniquely. Let p, q be continuous on an open interval I . For two solutions of $y'' + py' + qy = 0$, define

$$D(t_0) = u(t_0)v'(t_0) - u'(t_0)v(t_0), \quad t_0 \in I.$$

Tasks

1. Using existence and uniqueness for initial data, prove that the pair is fundamental on I if and only if $D(t_0) \neq 0$.
2. For $y'' + 2y' + 5y = 0$ on $\mathbb{R}$, verify $u = e^{-t} \cos 2t$, $v = e^{-t} \sin 2t$ and apply the criterion at zero.
3. Construct solutions ϕ, ψ with initial data $(\phi(0), \phi'(0)) = (1, 0)$ and $(\psi(0), \psi'(0)) = (0, 1)$.
4. Use this normalized pair to solve $y(0) = 2$, $y'(0) = -3$, and verify the data.

Question 1 – Solution

Strategy. A fundamental pair must supply every value-and-slope pair, so test the two-by-two initial-data system.

Step 1: Prove the criterion. If $D(t_0) \neq 0$, the equations

$$Au(t_0) + Bv(t_0) = a, \quad Au'(t_0) + Bv'(t_0) = b$$

have unique constants A, B for every a, b . The resulting combination solves the equation; uniqueness makes it equal to every solution with those data. Conversely, if $D(t_0) = 0$, some nonzero constant pair (A, B) gives zero value and slope. By uniqueness, $Au + Bv$ is identically zero, contradicting unique representation. Thus the criterion is necessary and sufficient.

Step 2: Verify the candidates. Their derivatives are

$$u' = e^{-t}(-\cos 2t - 2\sin 2t), \quad u'' = e^{-t}(-3\cos 2t + 4\sin 2t),$$

$$v' = e^{-t}(-\sin 2t + 2\cos 2t), \quad v'' = e^{-t}(-3\sin 2t - 4\cos 2t).$$

Both residuals vanish. At zero their data are $(1, -1)$ and $(0, 2)$, so $D(0) = 2 \neq 0$ and the pair is fundamental on $\mathbb{R}$.

Step 3: Normalize the data. The required choices are

$$\phi = u + \frac{1}{2}v, \quad \psi = \frac{1}{2}v.$$

Their data are $(1, 0)$ and $(0, 1)$ respectively, so they too form a fundamental set.

Step 4: Fit the solution. The normalized coefficients equal the data:

$$y = 2\phi - 3\psi = e^{-t}(2\cos 2t - \frac{1}{2}\sin 2t).$$

At zero its value is 2; its slope is $-2 - 1 = -3$. It satisfies the equation by linearity and is unique on $\mathbb{R}$.

Question 2

For the equation $y'' - y = 0$ on $\mathbb{R}$, consider the parameter-dependent pair

$$u = e^t + e^{-t}, \quad v_\alpha = e^t + \alpha e^{-t}, \quad \alpha \in \mathbb{R}.$$

Coefficients in a fundamental set must be constant in t .

Tasks

1. Verify both solutions and find all α for which the pair is fundamental.
2. For those parameters, find the coefficients of the solution with $y(0) = 0$, $y'(0) = 1$ in this pair.
3. As $\alpha \rightarrow 1$, compare the behavior of these coefficients with the behavior of the represented solution.
Can large coefficients alone establish large solution values?
4. Construct from u, v_α a companion with data $(0, 1)$ that extends to a well-defined function at $\alpha = 1$.
Explain why simply setting $\alpha = 1$ in the original pair fails.

Question 2 – Solution

Strategy. Track the initial-data vectors as the two proposed basis elements approach dependence.

Step 1: Test independence. Each exponential solves $y'' - y = 0$, so both combinations do. The data vectors at zero are $(2, 0)$ and $(1 + \alpha, 1 - \alpha)$, with determinant $2(1 - \alpha)$. The pair is fundamental exactly when $\alpha \neq 1$. At 1, both functions equal u .

Step 2: Fit the data. In $y = Au + Bv_\alpha$, the data require

$$2A + (1 + \alpha)B = 0, \quad (1 - \alpha)B = 1.$$

Thus

$$A = -\frac{1 + \alpha}{2(1 - \alpha)}, \quad B = \frac{1}{1 - \alpha}.$$

Step 3: Simplify before taking a limit. Substitution cancels the large terms exactly:

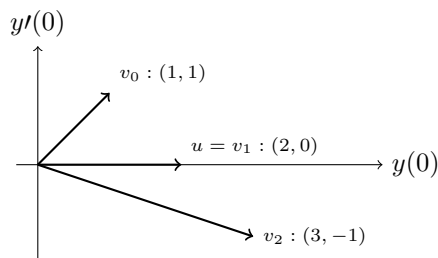
$$Au + Bv_\alpha = \frac{1}{2}(e^t - e^{-t}) = \sinh t.$$

Both coefficients grow without bound in magnitude near $\alpha = 1$, while the function is independent of α . Large coordinates in a nearly dependent pair do not imply large values of the function on a fixed interval.

Step 4: Replace the collapsing direction. Define, for $\alpha \neq 1$,

$$z_\alpha = \frac{v_\alpha - (1 + \alpha)u/2}{1 - \alpha} = \sinh t.$$

This extends by $z_1 = \sinh t$ and has data $(0, 1)$. The pair u, z_1 remains fundamental. The original pair at $\alpha = 1$ contains only one effective direction and cannot supply the requested nonzero slope.



Question 3

For $y'' - 3y' + 2y = 0$ on $\mathbb{R}$, let

$$f = e^t, \quad g = e^{2t}, \quad h = f + g, \quad k = 2f - g.$$

A student writes $y = Af + Bg + Ch + Dk$ and claims this gives four independent solution freedoms.

Tasks

1. Verify that all four functions solve the equation and that f, g form a fundamental set.
2. Determine which of the six unordered pairs chosen from f, g, h, k are fundamental. Justify every pair.
3. Find every quadruple (A, B, C, D) that represents the zero function, and explain the error in the student's claim.
4. Find all quadruples representing the solution with $y(0) = 1$, $y'(0) = 3$. Give its unique representation using only h, k .

Question 3 – Solution

Strategy. Express every candidate in the already verified pair f, g , then count effective coefficients.

Step 1: Verify the underlying pair. Substitution of e^{rt} gives residual $(r^2 - 3r + 2)e^{rt}$, which vanishes for $r = 1, 2$. Linearity verifies h, k . The data determinant for f, g at zero is $2 - 1 = 1$, so this pair is fundamental.

Step 2: Check all six pairs. The coefficient columns relative to f, g are $(1, 0)$, $(0, 1)$, $(1, 1)$ and $(2, -1)$. Their determinants, in the listed order, are

pair	(f, g)	(f, h)	(f, k)	(g, h)	(g, k)	(h, k)
determinant	1	1	-1	-1	-2	-3

Every determinant is nonzero, so every pair is an invertible change of the fundamental pair f, g and is itself fundamental.

Step 3: Identify redundancy. The proposed expression is

$$y = (A + C + 2D)f + (B + C - D)g.$$

It is zero exactly when $A = -C - 2D$ and $B = -C + D$, where C, D are arbitrary. These nontrivial constant relations show that the four functions are not independent. Only two effective coefficients are present.

Step 4: Fit the specified data. Writing $y = af + bg$ gives $a + b = 1$, $a + 2b = 3$, hence $a = -1, b = 2$. All four-coefficient representations are

$$(A, B, C, D) = (-1 - C - 2D, 2 - C + D, C, D).$$

Since $h - k = -f + 2g$, the unique representation in the pair h, k is $y = h - k$. Its data are $-1 + 2 = 1$ and $-1 + 4 = 3$.

Question 4

The smooth functions $u = t^2$ and $v = t^3$ are proposed as a fundamental set for an equation $y'' + p(t)y' + q(t)y = 0$ with continuous coefficients on $\mathbb{R}$.

Tasks

1. Prove that u, v are linearly independent as functions on $\mathbb{R}$.
2. Prove that no such regular normalized equation on $\mathbb{R}$ can have both as solutions. Explain why independence alone is insufficient.
3. Find the normalized equation having both as solutions on $(0, \infty)$ and verify that they form a fundamental set there.
4. On $(0, \infty)$, solve $y(1) = 0$, $y'(1) = 1$ using the pair. Explain why its zero value and slope at zero do not contradict your conclusion about regular equations.

Question 4 – Solution

Strategy. Check both requirements: independence and membership in the solution space of one regular equation on the stated interval.

Step 1: Prove independence. If $At^2 + Bt^3 = 0$ for every real t , then $A + Bt = 0$ for every nonzero t . Evaluating at two distinct nonzero points forces $A = B = 0$.

Step 2: Reject the proposed regular equation. The nonzero function $u = t^2$ has $u(0) = u'(0) = 0$. Uniqueness for continuous normalized coefficients would force it to equal the zero solution on $\mathbb{R}$. Equivalently, substitution at zero would require $u''(0) = 2 = 0$. Thus these independent functions cannot be a fundamental set for any equation of the stated kind.

Step 3: Work on a valid interval. Substituting the two functions for $t > 0$ gives

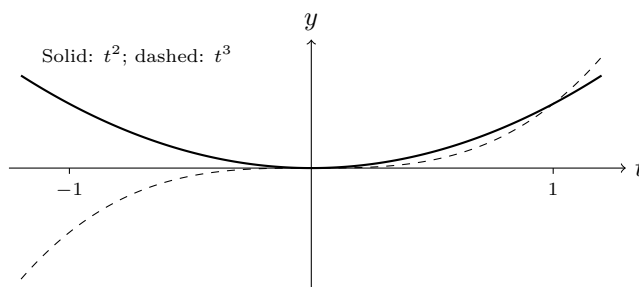
$$2 + 2tp + t^2q = 0, \quad 6t + 3t^2p + t^3q = 0.$$

Subtracting t times the first equation from the second yields $p = -4/t$, then $q = 6/t^2$. Hence

$$y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 0.$$

Both residuals vanish, the coefficients are continuous on $(0, \infty)$, and $uv' - u'v = t^4 > 0$ there, so the pair is fundamental.

Step 4: Fit and interpret. The data give $A + B = 0$, $2A + 3B = 1$, so $y = t^3 - t^2$. Its polynomial extension has zero value and slope at zero, but the normalized equation is undefined there. A theorem requiring continuous coefficients on an interval containing zero cannot be applied to that extension.



Question 5

The pair $\cos t, \sin t$ is fundamental for $y'' + y = 0$ on $\mathbb{R}$. Fix $L > 0$ and prescribe two position measurements

$$y(0) = a, \quad y(L) = b, \quad a, b \in \mathbb{R}.$$

Tasks

1. Find the coefficient equations and determine when the measurements select a unique solution.
2. For every exceptional length, give necessary and sufficient conditions for existence and describe all solutions when they exist.
3. At $L = \pi$, describe all solutions for $(a, b) = (1, -1)$ and decide whether any solution exists for $(a, b) = (1, 1)$.
4. Explain why the exceptional lengths do not mean that the fundamental pair has become dependent. Identify which additional measurement at zero would uniquely determine a solution.

Question 5 – Solution

Strategy. Distinguish a fundamental set of functions from the invertibility of a particular measurement map.

Step 1: Solve at ordinary lengths. Every solution is $A \cos t + B \sin t$. The position measurements give

$$A = a, \quad a \cos L + B \sin L = b.$$

If $\sin L \neq 0$, the unique solution is

$$y = a \cos t + \frac{b - a \cos L}{\sin L} \sin t.$$

Thus uniqueness holds exactly at lengths that are not positive integer multiples of π .

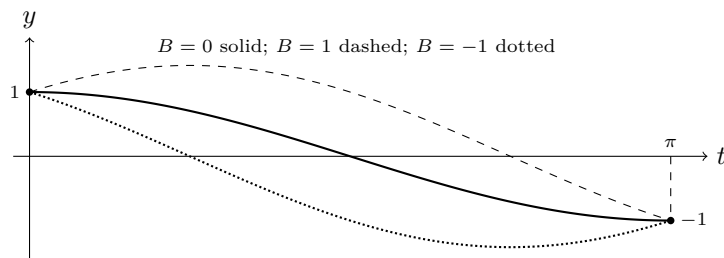
Step 2: Classify exceptional data. If $L = n\pi$, $n = 1, 2, \dots$, the second equation reduces to $b = (-1)^n a$. If this fails, there is no solution. If it holds, B is arbitrary and all solutions are $a \cos t + B \sin t$; there are infinitely many.

Step 3: Apply the classification. At $L = \pi$, the data $(1, -1)$ give

$$y = \cos t + B \sin t, \quad B \in \mathbb{R}.$$

The data $(1, 1)$ fail the necessary condition $b = -a$ and give no solution. The figure shows three members of the compatible family.

Step 4: Explain what failed. The data vectors of $\cos t, \sin t$ using value and slope at zero are $(1, 0)$ and $(0, 1)$, so the pair remains fundamental. At $L = n\pi$, both endpoint positions are insensitive to the coefficient B . Measuring $y'(0) = c$ fixes $B = c$ uniquely; the endpoint data must still be compatible with that solution.



Question 6

Let $D = \mathbb{R} \setminus \{0\}$. On this disconnected domain, consider $y'' = 0$ and the two functions

$$u(t) = 1, \quad v(t) = \begin{cases} -1, & t < 0, \\ 1, & t > 0. \end{cases}$$

A solution means a C^2 function on D ; no matching condition at zero is imposed.

Tasks

1. Verify that both functions solve the equation and are linearly independent on D .
2. Do their constant linear combinations represent every solution on D ? Give a decisive counterexample and compare their independence on each connected component.
3. Find the full solution family on D and an explicit independent spanning set. Prove why at least four functions are needed.
4. Describe every solution satisfying $y(1) = 0$, $y'(1) = 1$. Explain why those data do not determine a unique solution on all of D .

Question 6 – Solution

Strategy. Each connected component has its own integration constants because zero is excluded and no matching rule is imposed.

Step 1: Verify and test independence. Both functions are constant on each component, so both have zero second derivative on D . If $Au + Bv = 0$ on D , the positive side gives $A + B = 0$ and the negative side gives $A - B = 0$. Thus $A = B = 0$.

Step 2: Test spanning. Every combination $Au + Bv$ is constant on each side, so it cannot represent the solution $y = t$. The pair is independent on D but does not span its solution space. On either individual component, $v = u$ or $v = -u$, so the restricted pair is dependent.

Step 3: Exhibit all four freedoms. Integrating separately gives

$$y(t) = \begin{cases} A_- + B_-t, & t < 0, \\ A_+ + B_+t, & t > 0. \end{cases}$$

Let χ_+ equal 1 for $t > 0$ and 0 for $t < 0$, and let $\chi_- = 1 - \chi_+$. An independent spanning set is

$$\{\chi_-, t\chi_-, \chi_+, t\chi_+\}.$$

Restriction to either side proves independence by the independence of $1, t$. Each of the four displayed constants can be chosen freely, so the space has dimension four and no set with fewer than four functions spans it. All four functions are smooth on D .

Step 4: Impose data on one component. On the positive side, $B_+ = 1$ and $A_+ = -1$. Thus $y = t - 1$ for $t > 0$, while A_-, B_- remain arbitrary for $t < 0$. Initial-value uniqueness controls the connected interval containing the initial point; it supplies no information across the excluded point.

Question 7

Consider $y'' + 4y = 0$ on $\mathbb{R}$. Allow complex-valued solutions initially and define

$$z_+ = e^{2it}, \quad z_- = e^{-2it}, \quad i^2 = -1.$$

Then restrict attention to real-valued solutions.

Tasks

1. Verify the two complex solutions and show that they form a fundamental set over the complex numbers.
2. Find the necessary and sufficient relation between complex constants A, B for $Az_+ + Bz_-$ to be real for every real t .
3. Derive the correspondence with real coefficients in $a \cos 2t + b \sin 2t$. Explain why z_+, z_- themselves are not a fundamental set of the real-valued solution space.
4. Solve $y(0) = 1, y'(0) = -2$ in both representations and verify the coefficients.

Question 7 – Solution

Strategy. Specify the scalar field and impose real-valuedness on the entire function, not just its value at one point.

Step 1: Verify complex completeness. Each candidate has second derivative -4 times itself. Their data columns at zero are $(1, 2i)$ and $(1, -2i)$, with determinant $-4i \neq 0$. The initial-data criterion applies over $\mathbb{C}$ as well: real and imaginary parts each satisfy the real equation and its uniqueness theorem.

Step 2: Impose reality. Since $\overline{z_+} = z_-$, the conjugate of $Az_+ + Bz_-$ is $\overline{B}z_+ + \overline{A}z_-$. Independence implies equality to its conjugate exactly when

$$B = \overline{A}.$$

This relation is sufficient too, since the two terms then are conjugates.

Step 3: Convert coordinates. Euler's formula gives

$$Az_+ + Bz_- = (A + B) \cos 2t + i(A - B) \sin 2t.$$

For real a, b , the inverse relations are

$$A = (a - ib)/2, \quad B = (a + ib)/2.$$

They automatically satisfy the reality condition. The exponentials are not themselves real-valued functions on $\mathbb{R}$, so they are not elements of the real-valued solution space. A real fundamental set is $\cos 2t, \sin 2t$.

Step 4: Match the data. The real representation has $a = 1$ and $2b = -2$, hence

$$y = \cos 2t - \sin 2t = \frac{1}{2}(1 + i)e^{2it} + \frac{1}{2}(1 - i)e^{-2it}.$$

Here $A + B = 1$ and $2i(A - B) = -2$, verifying the value and slope. The conjugate coefficients ensure that the resulting solution is real everywhere.

Question 8

Two measured solution profiles are $u(t) = t$ and $v(t) = e^{-t}$. Seek a normalized equation

$$y'' + p(t)y' + q(t)y = 0$$

with both as solutions on a connected open interval containing $t = 0$.

Tasks

1. Use the two profiles to determine p, q wherever the coefficient equations are solvable.
2. Find the largest interval containing zero on which the reconstructed coefficients are continuous, and prove that the profiles form a fundamental set there.
3. Prove that no choice of continuous normalized coefficients on an interval containing -1 can have both profiles as solutions. Does their smoothness resolve this obstruction?
4. Use the fundamental pair to solve $y(0) = 2$, $y'(0) = -1$. Verify the data and explain the distinction between extending this formula and extending the normalized equation.

Question 8 – Solution

Strategy. Treat the two known solutions as simultaneous equations for the unknown coefficients, then check the interval where the result is regular.

Step 1: Reconstruct the equation. Substituting $u = t$ and $v = e^{-t}$ gives

$$p + tq = 0, \quad 1 - p + q = 0.$$

For $t \neq -1$, these have the unique solution

$$p = \frac{t}{t+1}, \quad q = -\frac{1}{t+1}.$$

Thus both residuals vanish in $y'' + [t/(t+1)]y' - y/(t+1) = 0$ wherever these coefficients are defined.

Step 2: Find a fundamental interval. The largest connected open interval containing zero and avoiding the pole is $(-1, \infty)$. The initial-data determinant, or its value at any point of this interval, is

$$uv' - u'v = -(t+1)e^{-t} \neq 0.$$

In particular it equals -1 at zero, so the two solutions form a fundamental set throughout the interval.

Step 3: Locate the obstruction. At $t = -1$, the coefficient equations would require $p - q = 0$ and $1 - p + q = 0$, an immediate contradiction. Even finite pointwise coefficient values cannot satisfy both conditions there. Smoothness of the two functions does not guarantee a common regular normalized equation across that point.

Step 4: Solve and distinguish extensions. In $y = At + Be^{-t}$, the data give $B = 2$ and $A - B = -1$, hence

$$y = t + 2e^{-t} \quad (-1 < t < \infty).$$

The value and slope at zero are $2, -1$, and linearity verifies its equation. The formula is smooth for all real t , but the reconstructed normalized equation is undefined at -1 . Extending a function does not remove that coefficient singularity. Multiplying by $t+1$ instead gives a different, undivided formulation with a vanishing leading coefficient there.

Question 9

For $y'' + y = 0$ on $\mathbb{R}$, let the state at a time s mean the ordered pair $(y(s), y'(s))$. Fix a real reference time s .

Tasks

1. Construct a fundamental pair ϕ_s, ψ_s with state data $(1, 0)$ and $(0, 1)$ at time s . Express any solution in that pair.
2. Derive explicit formulas for the state at $s + h$ in terms of the state (a, b) at s .
3. Show that moving the reference time by h and then k gives the same result as moving it by $h + k$, and that the move by $-h$ reverses the move by h . Prove that $y^2 + (y')^2$ is preserved.
4. For initial state $(3, 4)$ at zero, find the first positive time when the slope is zero and the value then. Describe the direction of travel in the state plane.

Question 9 – Solution

Strategy. Changing the reference time changes the coordinates in a normalized fundamental pair, while the represented solution remains the same.

Step 1: Normalize at the reference time. Set

$$\phi_s(t) = \cos(t - s), \quad \psi_s(t) = \sin(t - s).$$

Both solve the equation and have the required data. Their data determinant is 1, so any solution is $y(t) = a \cos(t - s) + b \sin(t - s)$, where $a = y(s)$ and $b = y'(s)$.

Step 2: Move the state. Evaluation and differentiation give

$$y(s + h) = a \cos h + b \sin h, \quad y'(s + h) = -a \sin h + b \cos h.$$

Step 3: Compose and reverse. Applying the same formulas with increment k yields first component

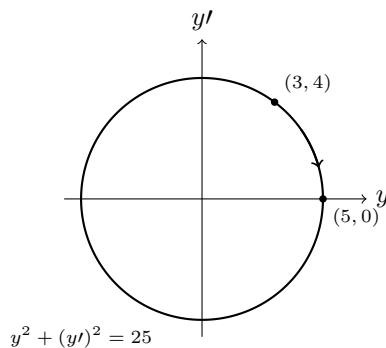
$$a(\cos h \cos k - \sin h \sin k) + b(\sin h \cos k + \cos h \sin k).$$

The addition formulas make this $a \cos(h + k) + b \sin(h + k)$; the second component is $-a \sin(h + k) + b \cos(h + k)$. Taking $k = -h$ gives (a, b) . Squaring the two components and adding cancels the cross terms and gives $a^2 + b^2$.

Step 4: Locate the first turning point. Here $y = 3 \cos t + 4 \sin t$ and $y' = -3 \sin t + 4 \cos t$. The first positive zero of y' is

$$t_* = \arctan(4/3), \quad y(t_*) = 5.$$

Indeed $\cos t_* = 3/5$, $\sin t_* = 4/5$, and the slope is positive before this time. In coordinates (y, y') , the state travels clockwise on the circle of radius 5: at $(5, 0)$ its derivative is $(0, -5)$.



Question 10

For $y'' - y = 0$ on $\mathbb{R}$, fix $L > 0$. Seek solutions ℓ, r with endpoint values

$$\ell(0) = 1, \quad \ell(L) = 0, \quad r(0) = 0, \quad r(L) = 1.$$

Use $\sinh x = (e^x - e^{-x})/2$ and $\cosh x = (e^x + e^{-x})/2$.

Tasks

1. Construct ℓ, r from the exponential fundamental pair and prove that they also form a fundamental set.
2. Use this pair to solve $y(0) = a, y(L) = b$ for arbitrary real a, b . Prove uniqueness.
3. For $a, b \geq 0$ and not both zero, prove positivity on $0 < t < L$ and $y(t) \leq \max(a, b)$ on $[0, L]$. Determine whether $\ell(t) + r(t) = 1$ in the interior.
4. For $a = b = 1$, locate the unique minimum on $[0, L]$ and give its value. Explain why positive interpolation weights do not imply straight-line interpolation.

Question 10 – Solution

Strategy. Choose basis functions adapted to the two measurements, then inspect their positive weights.

Step 1: Build an endpoint basis. Write $r = Ae^t + Be^{-t}$. The data give $A + B = 0$ and $A(e^L - e^{-L}) = 1$, so $A = 1/(2 \sinh L)$ and $B = -A$. Reflecting r about the midpoint gives $\ell(t) = r(L - t)$, hence

$$\ell(t) = \frac{\sinh(L - t)}{\sinh L}, \quad r(t) = \frac{\sinh t}{\sinh L}.$$

Both solve $y'' - y = 0$ and have the stated endpoint values. Their determinant at zero is $\ell(0)r'(0) - \ell'(0)r(0) = 1/\sinh L \neq 0$, so they form a fundamental set on $\mathbb{R}$.

Step 2: Fit the measurements. In this pair the endpoint coefficients are exactly the prescribed values:

$$y = a\ell + br.$$

Every solution has a unique representation, and evaluating at $0, L$ forces its coefficients to be a, b . Thus the endpoint problem has exactly one solution.

Step 3: Bound the positive weights. On $0 < t < L$, both weights are positive. The hyperbolic addition identity gives

$$\ell(t) + r(t) = \frac{\cosh(t - L/2)}{\cosh(L/2)} \leq 1 \quad (0 \leq t \leq L),$$

with strict inequality in the interior, since $|t - L/2| < L/2$. If $M = \max(a, b)$, then $0 < y$ in the interior and $y \leq M(\ell + r) \leq M$ on the full closed interval. At an endpoint the solution can equal zero if the corresponding datum is zero.

Step 4: Examine equal endpoint data. For $a = b = 1$,

$$y = \frac{\cosh(t - L/2)}{\cosh(L/2)}, \quad y_{\min} = \frac{1}{\cosh(L/2)} \text{ at } t = L/2.$$

The minimum is unique because $\cosh x$ has its unique minimum at zero. The weights are positive but sum to less than one inside; this is not the straight-line interpolation of equal endpoint values.

