

Question 1

Let p, q be continuous on an open interval I , let $t_0 \in I$, and let y_1 be a known solution of

$$y'' + p(t)y' + q(t)y = 0$$

that never vanishes on I . Write $P(t) = \int_{t_0}^t p(s) ds$.

Tasks

1. Derive the equation for $w = v'$ after substituting $y = y_1 v$.
2. Solve the first-order equation for w and give an integral formula for y .
3. Construct a solution z with $z(t_0) = 0$ and $z'(t_0) = 1$. Prove that z is not a constant multiple of y_1 .
4. Express the solution with $y(t_0) = a$, $y'(t_0) = b$ in terms of y_1, z . Explain why the nonvanishing hypothesis matters.

Question 1 – Solution

Strategy. Factor out the known solution, then normalize the new solution by its initial slope.

Step 1: Reduce the order. Product differentiation gives

$$y' = y_1'v + y_1v', \quad y'' = y_1''v + 2y_1'v' + y_1v''.$$

The coefficient of v is zero because y_1 solves the equation. Hence

$$y_1v'' + (2y_1' + py_1)v' = 0, \quad w' + (2y_1'/y_1 + p)w = 0.$$

Step 2: Integrate twice. An integrating factor is $y_1^2e^P$, so

$$w = C \frac{e^{-P}}{y_1^2}, \quad y = y_1(t) \left[A + C \int_{t_0}^t \frac{e^{-P(s)}}{y_1(s)^2} ds \right].$$

Both integrations contribute a constant. The formula includes $w = 0$.

Step 3: Normalize the companion. Set

$$z(t) = y_1(t_0)y_1(t) \int_{t_0}^t \frac{e^{-P(s)}}{y_1(s)^2} ds.$$

The integral vanishes at t_0 . Since $P(t_0) = 0$, differentiation gives $z'(t_0) = 1$. If $z = ky_1$, its zero value forces $k = 0$, contradicting this slope.

Step 4: Match arbitrary data. The two constants are fixed by

$$y = \frac{a}{y_1(t_0)}y_1 + \left[b - \frac{ay_1'(t_0)}{y_1(t_0)} \right]z.$$

This is the unique solution on I by the regular linear initial-value theorem. Division by y_1 and the integral kernel require $y_1 \neq 0$. A zero of the seed may obstruct this substitution without obstructing the original equation.

Question 2

On $t > 0$, consider

$$t^2 y'' - 3ty' + 4y = 0, \quad y(1) = 1, \quad y'(1) = 3.$$

A proposed known solution is $y_1 = t^2$.

Tasks

1. Verify y_1 and use reduction of order to find the general solution.
2. Solve the initial-value problem and verify both data.
3. Find every zero and stationary point of this solution on $t > 0$, and classify the stationary point.
4. Determine whether this solution has a C^1 extension and whether it has a C^2 extension through $t = 0$. Justify using one-sided limits.

Question 2 – Solution

Strategy. Normalize the equation before reduction, then use exact derivatives to study the singular endpoint.

Step 1: Find the missing solution. For t^2 , the residual is $(2 - 6 + 4)t^2 = 0$. With $y = t^2v$, the divided equation reduces to

$$v'' + \frac{1}{t}v' = 0, \quad v' = D/t, \quad \boxed{y = t^2(C + D \ln t)}.$$

Step 2: Impose the data. At 1, $y = C$ and $y' = 2C + D$. Thus $C = D = 1$ and

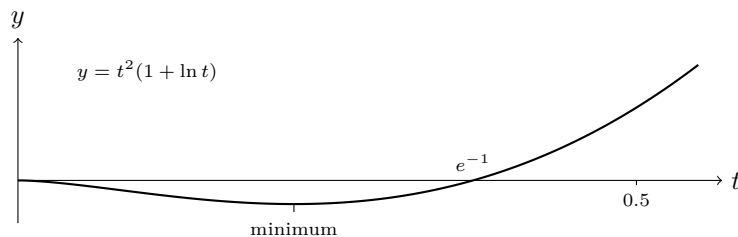
$$\boxed{y = t^2(1 + \ln t)}, \quad y' = t(3 + 2 \ln t), \quad y'' = 5 + 2 \ln t.$$

These give $y(1) = 1$ and $y'(1) = 3$.

Step 3: Locate the features. The sole zero is $t = e^{-1}$. The derivative changes from negative to positive at $t = e^{-3/2}$, giving the unique global minimum

$$(t, y) = (e^{-3/2}, -\frac{1}{2}e^{-3}).$$

Step 4: Test endpoint regularity. As $t \downarrow 0$, both y and y' tend to zero, but $y'' \rightarrow -\infty$. Setting $y = 0$ for $t \leq 0$ gives a C^1 extension (the derivative at zero is also $\lim y(t)/t = 0$). No C^2 extension is possible. The leading coefficient vanishes at zero, so the regular normalized theorem does not apply there.



Question 3

Consider the equation on $\mathbb{R}$

$$y'' - 2ty' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 1,$$

with known solution $y_1 = e^{t^2}$. An integral is an acceptable exact answer.

Tasks

1. Verify the seed and derive the reduced first-order equation.
2. Find the initial-value solution as a definite integral and verify its residual without evaluating the integral.
3. Prove that the solution is odd and strictly increasing on $\mathbb{R}$.
4. Prove $t \leq y(t) \leq te^{t^2}$ for $t \geq 0$ and compute $y(1)$ to six decimal places. Explain why an unevaluated integral still defines a valid solution.

Question 3 – Solution

Strategy. Keep the definite integral and use the fundamental theorem of calculus for verification and bounds.

Step 1: Reduce. The seed has derivatives $2te^{t^2}$ and $(2 + 4t^2)e^{t^2}$, so its residual vanishes. With $y = e^{t^2}v$,

$$w' + 2tw = 0, \quad w = v' = Ce^{-t^2}.$$

Step 2: Fit and verify. The zero initial value sets the constant in v to zero; the initial slope sets $C = 1$. Therefore

$$y(t) = e^{t^2} \int_0^t e^{-s^2} ds.$$

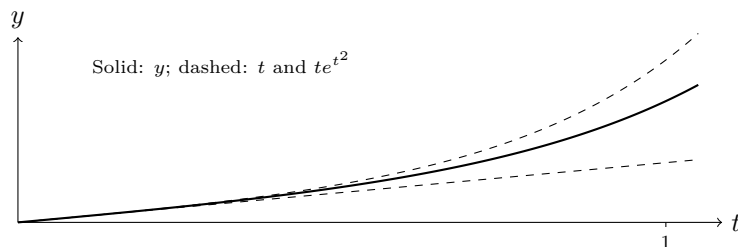
Differentiation gives $y' = 2ty + 1$, then $y'' = 2y + 2ty'$. The residual is zero, and at zero the value and slope are 0, 1.

Step 3: Use symmetry and sign. The integrand is positive and even, so its integral from zero is odd and has the sign of t . Multiplication by e^{t^2} preserves both properties. Consequently $ty \geq 0$ and $y' = 2ty + 1 \geq 1$ everywhere: y is strictly increasing.

Step 4: Bound and evaluate. For $0 \leq s \leq t$,

$$e^{-t^2} \leq e^{-s^2} \leq 1 \implies te^{-t^2} \leq \int_0^t e^{-s^2} ds \leq t.$$

Multiplying by e^{t^2} proves the bounds. Numerical integration gives $y(1) \approx 2.030078$. The integrand is smooth, so the definite integral defines a smooth function on all of $\mathbb{R}$; the exact residual check proves it solves the equation.



Question 4

For $y'' + y = 0$, suppose the known solution is $y_1 = \sin t$. First work on $(0, \pi)$, where the seed does not vanish.

Tasks

1. Use $y = (\sin t)v$ to construct a second solution on $(0, \pi)$.
2. Explain whether the poles of the quotient v prevent the second solution from extending to $\mathbb{R}$.
3. Use the extended solutions to solve $y(0) = 1$, $y'(0) = 0$, and justify uniqueness.
4. A proposed continuation is $\cos t + a \sin t$ for $t < 0$ and $\cos t + b \sin t$ for $t > 0$, with $y(0) = 1$. Find the conditions for continuity, for C^1 regularity, and for being a classical solution through zero.

Question 4 – Solution

Strategy. Separate a failure of division by the seed from a singularity of the original equation.

Step 1: Reduce on the valid interval. Substitution gives

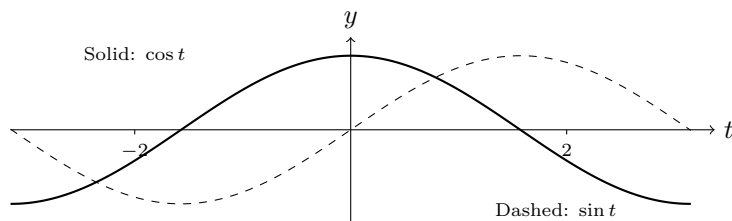
$$v'' + 2 \cot t v' = 0, \quad v' = C \csc^2 t, \quad v = -C \cot t + D.$$

Thus $y = -C \cos t + D \sin t$; a second solution is $\cos t$.

Step 2: Extend the product. Although $\cos t / \sin t = \cot t$ has poles at multiples of π , its product with $\sin t$ is $\cos t$ on each valid interval. The latter is smooth and directly satisfies $y'' + y = 0$ on all of $\mathbb{R}$. The poles belong to the chosen quotient.

Step 3: Solve at the seed's zero. In $y = A \cos t + B \sin t$, the data give $A = 1$, $B = 0$. Thus $\boxed{y = \cos t}$. Continuous coefficients and nonzero leading coefficient ensure global uniqueness for the data at zero.

Step 4: Match derivatives. Both one-sided values are 1, so continuity holds for all real a, b . The one-sided slopes are a, b , so C^1 regularity requires $a = b$. In that case one smooth formula holds on both sides, and it is a classical solution. If $a \neq b$, a derivative at zero does not exist; matching only position is insufficient.



Question 5

On $t > 0$, consider $t^2 y'' - 2ty' + 2y = 0$ with seed $y_1 = t$. Use the following *unnormalized* companion convention:

$$Y_a(t) = y_1(t) \int_a^t \frac{s^2}{y_1(s)^2} ds, \quad a > 0.$$

The numerator s^2 is held fixed when changing a or rescaling the seed.

Tasks

1. Verify the seed and derive the kernel in this convention from the normalized equation.
2. Compute Y_1 , Y_2 , and the companion obtained with seed $3t$ and lower limit 1.
3. Explain why these choices give the same solution space. Why does an arbitrary additive integration constant not create a third solution freedom?
4. Use $y = At + BY_1(t)$ to solve $y(1) = 2$, $y'(1) = 1$. Verify the equation and data.

Question 5 – Solution

Strategy. Compare the actual functions produced by changes in normalization, rather than counting written constants.

Step 1: Identify the kernel. The seed residual is $-2t + 2t = 0$. After division by t^2 , $p = -2/t$ and

$$P(t) = \int_1^t -2/s \, ds = -2 \ln t, \quad e^{-P(t)} = t^2.$$

Reduction gives $v' = Ct^2/y_1^2$, which yields the specified convention.

Step 2: Compute three companions. For $y_1 = t$, the integrand equals 1, so

$$Y_1 = t(t-1) = t^2 - t, \quad Y_2 = t(t-2) = t^2 - 2t.$$

For seed $3t$, the integrand is $1/9$, hence

$$\tilde{Y}_1 = 3t(t-1)/9 = (t^2 - t)/3.$$

Step 3: Compare freedoms. We have $Y_2 = Y_1 - t$ and $\tilde{Y}_1 = Y_1/3$. Adding a seed multiple or multiplying a companion by a nonzero constant leaves the family unchanged: every choice spans $\{At + Bt^2 : A, B \in \mathbb{R}\}$. An additive constant inside the antiderivative produces only another multiple of the seed, already present in the family.

Step 4: Fit and verify. Since $Y_1(1) = 0$, $Y_1'(1) = 1$, the data give $A = 2$ and $A + B = 1$. Therefore

$$y = 2t - (t^2 - t) = 3t - t^2.$$

Its derivatives are $3 - 2t$ and -2 , and

$$t^2(-2) - 2t(3 - 2t) + 2(3t - t^2) = 0.$$

At $t = 1$ the value is 2 and the slope is 1. Two effective constants are sufficient and necessary for arbitrary regular initial data.

Question 6

On $t > 0$, consider

$$ty'' - (t+1)y' + y = 0, \quad y_1 = e^t.$$

A student inserts the undivided coefficient $-(t+1)$ into the usual normalized reduction formula and obtains $v' = e^{t^2/2-t}$.

Tasks

1. Verify the seed, normalize the equation, and derive the correct equation for $w = v'$.
2. Find the full solution family and solve $y(1) = 0$, $y'(1) = 1$.
3. Compute the residual of the student's candidate $y = e^t v$, with $v' = e^{t^2/2-t}$. Explain why testing only $t = 1$ would miss the error.
4. For the original undivided equation at $t = 0$, find the necessary relation between value and slope. Exhibit two distinct smooth solutions with zero value and slope there.

Question 6 – Solution

Strategy. The coefficient in the standard formula is the coefficient after division by the leading term.

Step 1: Normalize. For e^t , the residual is $[t - (t + 1) + 1]e^t = 0$. Division gives $p = -1 - 1/t$, so

$$w' + (1 - 1/t)w = 0, \quad w = Cte^{-t}.$$

Step 2: Integrate and fit. Since $\int te^{-t} dt = -(t + 1)e^{-t}$,

$$y = Ae^t + B(t + 1).$$

At 1 the data give $Ae + 2B = 0$ and $Ae + B = 1$. Thus $B = -1$, $A = 2/e$, and

$$\boxed{y = 2e^{t-1} - (t + 1)}.$$

Both basis functions have zero residual, and substitution at 1 gives 0, 1.

Step 3: Diagnose the error. For any v , the undivided residual of $e^t v$ is

$$e^t[tw' + (t - 1)w].$$

The student's w satisfies $w' = (t - 1)w$, giving residual

$$e^t(t^2 - 1)e^{t^2/2-t}.$$

This is zero at $t = 1$ but equals $3e^2$ at $t = 2$. A single vanishing residual does not establish a differential identity.

Step 4: Examine the singular initial point. At zero the original equation requires $y'(0) = y(0)$. Both $y = 0$ and $y = e^t - (t + 1)$ are smooth solutions through zero with value and slope zero; the second is nonzero, for example because its second derivative at zero is 1. The regular initial-value theorem fails here because the leading coefficient vanishes. Compatibility does not imply uniqueness.

Question 7

On $\mathbb{R}$, consider $(1+t^2)y'' - 2y = 0$ with known solution $y_1 = 1+t^2$. Let y_2 denote the solution with $y_2(0) = 0$, $y_2'(0) = 1$, and define $I = y_2/(1+t^2)$.

Tasks

1. Use reduction of order to find y_2 explicitly. Evaluate the required integral.
2. Verify the equation and the two normalization data directly.
3. Find the exact range and monotonicity of $I(t)$ on $\mathbb{R}$, including the limiting values at infinity.
4. For every real C , classify the finite zeros and the sign of $y_C = C(1+t^2) - y_2$. Include the two boundary parameter values.

Question 7 – Solution

Strategy. Divide the solution family by its positive seed so that zeros become level crossings of a monotone function.

Step 1: Construct the companion. The seed residual is $2(1+t^2) - 2(1+t^2) = 0$. Since $p = 0$,

$$I(t) = \int_0^t \frac{ds}{(1+s^2)^2} = \frac{t}{2(1+t^2)} + \frac{1}{2} \arctan t.$$

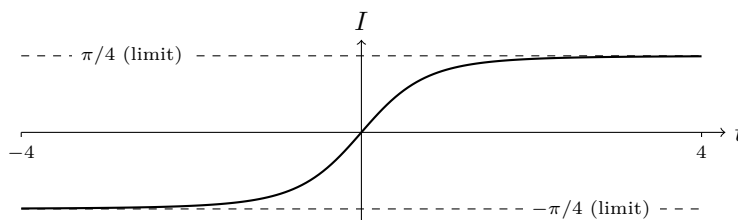
The integral follows by setting $s = \tan u$ and integrating $\cos^2 u$. Hence

$$y_2 = \frac{1}{2}t + \frac{1}{2}(1+t^2) \arctan t.$$

Step 2: Verify. We obtain $y_2' = 1 + t \arctan t$ and $y_2'' = \arctan t + t/(1+t^2) = 2y_2/(1+t^2)$. The equation holds and the data are 0, 1.

Step 3: Determine the ratio range. Since $I' = 1/(1+t^2)^2 > 0$, the ratio is strictly increasing. Its limits are $-\pi/4$ and $\pi/4$; neither is attained. Its range is exactly $(-\pi/4, \pi/4)$ by continuity.

Step 4: Classify all parameters. The sign of y_C is the sign of $C - I(t)$. If $|C| < \pi/4$, there is exactly one finite zero, with positive values before it and negative values after it. If $C \geq \pi/4$, the solution is strictly positive everywhere; if $C \leq -\pi/4$, it is strictly negative everywhere. Equality in these last two cases does not create a finite zero because the limiting ratio levels are excluded.



Question 8

On $\mathbb{R}$, solve the boundary-at-infinity problem

$$y'' + \frac{2t}{1+t^2}y' = 0, \quad \lim_{t \rightarrow -\infty} y(t) = -1, \quad \lim_{t \rightarrow \infty} y(t) = 3.$$

The constant function $y_1 = 1$ is a known homogeneous solution.

Tasks

1. Use reduction of order to obtain the full solution family.
2. Find the unique solution satisfying the two limiting conditions and verify its equation.
3. Find its value and slope at zero, monotonicity and exact range. Is every solution of this equation bounded?
4. Compute $\lim_{t \rightarrow \infty} t[3 - y(t)]$. Decide whether constants $M, c > 0$ can satisfy $3 - y(t) \leq Me^{-ct}$ for all sufficiently large t .

Question 8 – Solution

Strategy. A constant seed reduces the equation directly to a first-order equation for the slope.

Step 1: Integrate the slope. Set $w = y'$. An integrating factor is $1 + t^2$, so

$$((1 + t^2)w)' = 0, \quad w = B/(1 + t^2), \quad y = A + B \arctan t.$$

Step 2: Fit the two limits. The equations $A - B\pi/2 = -1$ and $A + B\pi/2 = 3$ give

$$y = 1 + \frac{4}{\pi} \arctan t.$$

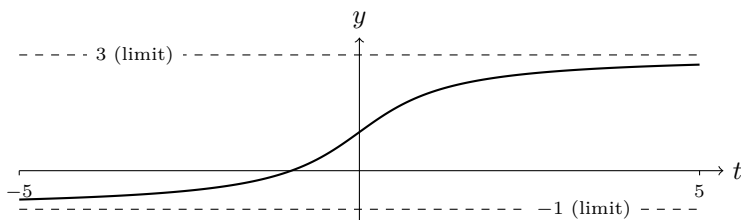
This is the unique choice of the two constants. Its derivatives $4/[\pi(1 + t^2)]$ and $-8t/[\pi(1 + t^2)^2]$ cancel in the original equation.

Step 3: Describe the trajectory. The value and slope at zero are 1 and $4/\pi$. The slope is positive everywhere; continuity and the limiting values give range $(-1, 3)$. Every member $A + B \arctan t$ is bounded, since $|\arctan t| < \pi/2$.

Step 4: Determine the tail rate. For $t > 0$,

$$3 - y(t) = \frac{4}{\pi} \arctan(1/t), \quad \lim_{t \rightarrow \infty} t[3 - y(t)] = \frac{4}{\pi}.$$

Here $\arctan u/u \rightarrow 1$ as $u \rightarrow 0$. An exponential bound would force this positive limit to be at most $\lim Mte^{-ct} = 0$, a contradiction. The approach to the limit has an algebraic $1/t$ tail.



Question 9

Consider the original undivided equation $t^2 y'' - 6y = 0$. First work on $t > 0$ with known solution $y_1 = t^3$. A classical solution through zero means a C^2 function satisfying the undivided equation on an open interval containing zero.

Tasks

1. Use reduction of order to find every solution on $t > 0$.
2. Which right-hand solutions admit a C^2 extension through zero? Find the forced value, slope and second derivative at zero.
3. For each admissible right-hand solution, classify all classical continuations to $t < 0$. Explain whether the zero initial data determine a unique continuation.
4. Which of those continuations are C^3 at zero? Explain why the usual uniqueness theorem does not settle this problem.

Question 9 – Solution

Strategy. Solve on each side of the singular coefficient, then compare one-sided derivatives at the join.

Step 1: Reduce on the positive side. The seed has residual $t^2(6t) - 6t^3 = 0$. In normalized form $p = 0$, so

$$v' = Ct^{-6}, \quad v = D - \frac{1}{5}Ct^{-5}, \quad \boxed{y = At^3 + Bt^{-2} \quad (t > 0)}.$$

These two powers each satisfy the original equation by direct substitution.

Step 2: Exclude the divergent mode. If $B \neq 0$, then y diverges at zero, so even a continuous extension is impossible. If $B = 0$, the limits of y, y', y'' are $0, 0, 0$. These are the required values for a C^2 extension.

Step 3: Classify continuations. On $t < 0$ the same calculation gives $at^3 + bt^{-2}$; continuity forces $b = 0$. Thus all classical continuations of the right branch At^3 are

$$y(t) = \begin{cases} at^3, & t < 0, \\ 0, & t = 0, \\ At^3, & t > 0, \end{cases} \quad a \in \mathbb{R}.$$

The first and second derivatives match at zero for every a . At zero the undivided residual is $0 - 6(0) = 0$. Therefore these are sufficient as well as necessary, and infinitely many continuations share $y(0) = y'(0) = 0$ even with the right branch fixed.

Step 4: Require one more derivative. The one-sided third derivatives are $6a$ and $6A$; a C^3 join occurs exactly when $a = A$. In that case the single polynomial At^3 holds on both sides. The leading coefficient is zero at the joining point, and the normalized coefficient $-6/t^2$ is undefined there, so the regular uniqueness theorem has no applicable interval containing zero.

Question 10

On $t > 0$, consider the forced equation

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = t, \quad y(1) = 0, \quad y'(1) = 0.$$

The function $y_1 = t$ is supplied as a solution of the *associated homogeneous equation*.

Tasks

1. Verify the stated role of y_1 and explain why it is not a solution of the forced equation.
2. Substitute $y = tv$ and derive a reduced equation. Find the full forced solution family, retaining both integration constants.
3. Solve the given initial-value problem.
4. Check the original residual and data directly. Explain the error in applying the homogeneous reduction formula without carrying along the forcing term.

Question 10 – Solution

Strategy. The known homogeneous solution still cancels the term containing v , but the nonzero right-hand side remains.

Step 1: Verify the seed's role. For $y_1 = t$, the left-hand side is $0 - 2/t + 2/t = 0$. This satisfies the homogeneous equation, but cannot equal the positive forcing t on the specified domain.

Step 2: Reduce with the forcing present. With $y = tv$,

$$y' = v + tv', \quad y'' = 2v' + tv''.$$

Substitution cancels the terms in v and v' :

$$2v' + tv'' - \frac{2}{t}(v + tv') + \frac{2}{t}v = tv'' = t.$$

Since $t > 0$, $v'' = 1$. Equivalently the first-order reduced equation is $w' = 1$, with $w = v'$. Integrating twice gives

$$v = \frac{1}{2}t^2 + At + B, \quad \boxed{y = \frac{1}{2}t^3 + At^2 + Bt.}$$

Step 3: Impose the data. At 1 the equations are $1/2 + A + B = 0$ and $3/2 + 2A + B = 0$. Thus $A = -1$, $B = 1/2$, yielding

$$\boxed{y = \frac{1}{2}t(t-1)^2.}$$

Step 4: Verify and diagnose. In expanded form the solution is $t^3/2 - t^2 + t/2$, with $y' = 3t^2/2 - 2t + 1/2$ and $y'' = 3t - 2$. Its left-hand side is

$$(3t - 2) + (-3t + 4 - 1/t) + (t - 2 + 1/t) = t.$$

At 1 its value and slope are zero. Discarding the forcing would instead give $w' = 0$, producing only $At^2 + Bt$, whose left-hand side is zero. Those two terms provide the homogeneous freedom, but the cubic term is essential to match the forcing.