

Question 1

Consider

$$4y'' - 12y' + 9y = 0, \quad y(2) = -1, \quad y'(2) = 1/2.$$

A student proposes $y = C_1 e^{3t/2} + C_2 e^{3t/2}$ as the general solution because the characteristic root occurs twice.

Tasks

1. Find the characteristic root and explain precisely which initial-data freedom the proposed family lacks.
2. Put $x = t - 2$ and $y = e^{3x/2}u(x)$. Differentiate to reduce this particular equation to an equation for u , and obtain the full solution family.
3. Fit the initial data and verify the resulting solution in the original equation and both data.
4. Show that the derived family can represent arbitrary value and slope at $t = 2$. Justify completeness and explain why multiplying one exponential by $t - 2$ creates a new freedom while merely naming a second coefficient does not.

Question 1 – Solution

Strategy. Remove the common exponential in this constant-coefficient equation and determine what equation remains for its multiplier.

Step 1: Identify the repeated root and lost freedom. The characteristic polynomial is $(2r - 3)^2$, so $r = 3/2$ is repeated. The student's family equals $(C_1 + C_2)e^{3t/2}$ and has only one effective constant. It always satisfies $y' = (3/2)y$, so at $t = 2$ a value of -1 would force slope $-3/2$, not $1/2$.

Step 2: Derive the missing multiplier. With $x = t - 2$ and $y = e^{3x/2}u$,

$$y' = e^{3x/2}(u' + 3u/2), \quad y'' = e^{3x/2}(u'' + 3u' + 9u/4).$$

Substitution gives $4y'' - 12y' + 9y = 4e^{3x/2}u''$. Thus $u'' = 0$ and $u = A + Bx$, yielding

$$y = (A + B(t - 2))e^{3(t-2)/2}.$$

Step 3: Apply and verify the data. At $t = 2$, $y = A$ and $y' = B + 3A/2$. Hence $A = -1$, $B = 2$, so

$$y = (-1 + 2(t - 2))e^{3(t-2)/2}.$$

The multiplier has zero second derivative, making the original residual zero by Step 2. At 2, the value is -1 and the derivative is $2 - 3/2 = 1/2$.

Step 4: Establish the two independent data freedoms. For arbitrary data $y(2) = a$, $y'(2) = b$, we obtain $A = a$, $B = b - 3a/2$, uniquely. The transformation is invertible because the exponential never vanishes, and integrating $u'' = 0$ gives all multipliers. Alternatively, continuous-coefficient IVP uniqueness proves completeness after the data map is verified.

The function $(t - 2)e^{3(t-2)/2}$ has zero value but nonzero derivative at the base point, allowing slope to change without changing the prescribed value. A second copy of the same exponential cannot do that.

Question 2

Consider

$$y'' + 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = -3.$$

All qualitative conclusions below concern $t \geq 0$.

Tasks

1. Solve the IVP using the repeated-root family and verify both data.
2. Find every nonnegative zero and determine the sign on either side. Does a repeated negative root prevent the response from changing sign?
3. Find every positive-time stationary point, classify it and calculate its value. Determine the limit and the side from which it is approached.
4. Sketch the response, marking the zero and minimum. Explain why the extra polynomial factor can change the shape without defeating eventual exponential decay.

Question 2 – Solution

Strategy. Use the nonvanishing exponential to separate zeros of the affine multiplier from zeros of the derivative.

Step 1: Fit the repeated-root solution. The characteristic equation is $(r+2)^2 = 0$, so $y = (A+Bt)e^{-2t}$. The data give $A = 1$, $B - 2A = -3$, hence $B = -1$. Therefore

$$y = (1 - t)e^{-2t}.$$

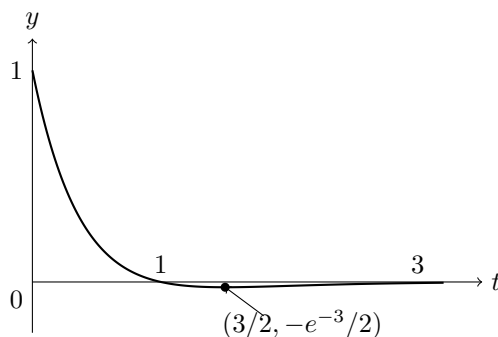
Its derivative is $(2t - 3)e^{-2t}$, with value -3 at zero, and substitution verifies $y'' + 4y' + 4y = 0$.

Step 2: Locate the sign change. Since $e^{-2t} > 0$, the sign is that of $1 - t$. The unique nonnegative zero is $t = 1$, with positive values before and negative values after. At the zero, $y'(1) = -e^{-2} \neq 0$, so this is a crossing rather than a tangency.

Step 3: Locate the minimum and the limiting behavior. The derivative vanishes only at $t = 3/2$. It is negative before and positive afterward, so the point is the unique global minimum on $[0, \infty)$:

$$y(3/2) = -\frac{1}{2}e^{-3}.$$

The factor t grows more slowly than e^{2t} , so $y \rightarrow 0$ as $t \rightarrow \infty$, approaching from below.



One crossing, then a minimum and approach from below.

Step 4: Interpret the roles of the two factors. The negative repeated root ensures decay of both e^{-2t} and te^{-2t} . It does not determine the sign of their combination or prevent a finite-time crossing. The affine multiplier supplies that sign change, while the derivative's affine multiplier determines a later turning point. The response crosses zero at 1, reaches its minimum at $3/2$, and then rises toward zero without crossing again.

Question 3

For $\lambda > 0$ and $a > 0$, consider

$$y'' + 2\lambda y' + \lambda^2 y = 0, \quad y(0) = a, \quad y'(0) = b.$$

Classify all real slopes b on $t \geq 0$.

Tasks

1. Find the solution in terms of a, b, λ .
2. Prove exactly when the response remains nonnegative for every $t \geq 0$.
3. Prove exactly when it is both nonnegative and nonincreasing, treating the boundary slopes separately.
4. For the remaining slopes, find the unique zero or positive-time maximum, as appropriate. Summarize the cases on a diagram of $b/(\lambda a)$ and explain how the repeated root enters the classification.

Question 3 – Solution

Strategy. Reduce both sign and monotonicity to affine functions of time after factoring out a positive exponential.

Step 1: Resolve the data. The root is $-\lambda$ twice. Writing $y = (A + Bt)e^{-\lambda t}$ gives $A = a$, $B = b + \lambda a$. Thus

$$y = (a + (b + \lambda a)t)e^{-\lambda t}.$$

For brevity let $B = b + \lambda a$.

Step 2: Classify nonnegativity. The multiplier starts at $a > 0$. It stays positive at every finite nonnegative time exactly when $B \geq 0$; if $B < 0$, it crosses zero and becomes negative. Therefore

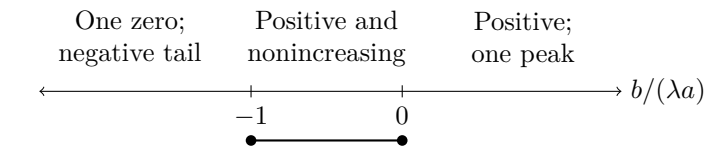
$$y(t) \geq 0 \text{ for all } t \geq 0 \iff b \geq -\lambda a.$$

At equality, the polynomial factor is constant and $y = ae^{-\lambda t}$.

Step 3: Impose nonincrease as well. Differentiation gives $y' = e^{-\lambda t}(b - \lambda Bt)$. With $B \geq 0$, this derivative is nonpositive for all $t \geq 0$ exactly when $b \leq 0$. Hence

$$-\lambda a \leq b \leq 0.$$

At $b = -\lambda a$, the response is a decreasing pure exponential. At $b = 0$, it has zero initial slope but negative derivative for every $t > 0$ because $B = \lambda a > 0$.



Both boundary slopes are included in the middle case.

Step 4: Classify the exterior ranges. If $b < -\lambda a$, then $B < 0$, and the unique zero is

$$t_z = -\frac{a}{b + \lambda a} > 0.$$

The response crosses from positive to negative and tends to zero from below. If $b > 0$, then $B > 0$ and the unique maximum occurs at

$$t_m = \frac{b}{\lambda(b + \lambda a)} > 0.$$

Here y' changes from positive to negative, and the response remains positive. The repeated root produces an affine multiplier; the zeros of that multiplier and its differentiated counterpart govern these thresholds. All cases still decay to zero.

Question 4

An unknown monic homogeneous second-order equation has a repeated real root r . A solution has exact samples

$$S_n = y(n), \quad S_0 = 1, \quad S_1 = 4/e, \quad S_2 = 7/e^2.$$

Write $y(t) = (A + Bt)e^{rt}$ and set $z = e^r > 0$.

Tasks

1. Derive the recurrence $S_{n+2} = 2zS_{n+1} - z^2S_n$.
2. Determine every positive z compatible with the three observations, then find the corresponding r, A, B . Explain why the observations do not yet identify a unique equation.
3. An additional sample gives $S_3 = 10/e^3$. Determine which candidate survives and recover the monic equation and initial slope.
4. Verify all four observations for the selected solution and calculate the rejected candidate's prediction at time 3. Contrast their long-time behavior and explain why three exact observations were insufficient here.

Question 4 – Solution

Strategy. Use the repeated-root sample recurrence, but retain every admissible root of the nonlinear identification equation.

Step 1: Derive the recurrence. Since $S_n = (A + Bn)z^n$,

$$2zS_{n+1} - z^2S_n = (2(A + B(n+1)) - (A + Bn))z^{n+2} = S_{n+2}.$$

Step 2: Find both admissible models. The observed values imply $7/e^2 = 8z/e - z^2$, or

$$(z - 1/e)(z - 7/e) = 0.$$

Both roots are positive. Since $A = S_0 = 1$ and $B = S_1/z - A$, the candidates are

z	r	A	B
$1/e$	-1	1	3
$7/e$	$\ln 7 - 1$	1	$-3/7$

Each gives a different repeated-root equation $y'' - 2ry' + r^2y = 0$, and each fits the original three observations. Selecting only one positive root of the identification quadratic would be unjustified.

Step 3: Use the fourth observation. For the first candidate, $S_3 = (1 + 9)e^{-3} = 10/e^3$, matching the new datum. The other gives

$$S_3 = (1 - 9/7)(7/e)^3 = \boxed{-98/e^3},$$

so it is rejected. The identified equation and data are

$$\boxed{y'' + 2y' + y = 0}, \quad \boxed{y = (1 + 3t)e^{-t}}, \quad \boxed{y'(0) = 2}.$$

Step 4: Verify and interpret the ambiguity. At $n = 0, 1, 2, 3$, the selected multiplier $1 + 3n$ gives 1, 4, 7, 10, with the required exponential factors. This solution tends to zero. The rejected candidate has $r = \ln 7 - 1 > 0$ and negative leading multiplier coefficient, so it eventually tends to $-\infty$.

The initial three sample constraints lead to a quadratic equation with two admissible positive multipliers, despite being exact. The fourth observation distinguishes them. Counting unknown parameters and observations alone does not prove a nonlinear inverse problem has a unique solution.

Question 5

For a real parameter r , consider

$$y'' - 2ry' + r^2y = 0.$$

A solution is forward bounded if bounded on $[0, \infty)$ and two-sided bounded if bounded on all of $\mathbb{R}$.

Tasks

1. Give the complete solution family for every real r , including $r = 0$.
2. Classify exactly which solutions are forward bounded and which tend to zero as $t \rightarrow \infty$, for each sign of r .
3. Classify all two-sided bounded solutions. Justify why a polynomial factor cannot cancel exponential growth in the growing time direction.
4. Explain why a repeated zero root behaves differently from a single zero root paired with a negative root. Give a counterexample to the statement that nonpositive characteristic roots always make every solution bounded.

Question 5 – Solution

Strategy. Treat the zero root separately and distinguish decay in one time direction from boundedness in both directions.

Step 1: Retain both repeated-root freedoms. For every r , the family is $y = (A + Bt)e^{rt}$. At $r = 0$ this becomes the affine family $y = A + Bt$ solving $y'' = 0$, not merely the constant family.

Step 2: Classify forward behavior. For $r < 0$, both e^{rt} and te^{rt} tend to zero, so every solution decays and is forward bounded. For $r = 0$, boundedness requires $B = 0$, while decay to zero requires $A = B = 0$. For $r > 0$, only the zero solution is forward bounded or tends to zero.

r	forward-bounded solutions	solutions tending to zero
$r < 0$	all solutions	all solutions
$r = 0$	$y = A$	$y = 0$
$r > 0$	$y = 0$	$y = 0$

For negative r , boundedness follows from the finite limit and continuity on every finite initial interval.

Step 3: Inspect the other time direction. If $B \neq 0$, the magnitude of $A + Bt$ is at least $|B||t|/2$ for sufficiently large $|t|$. If $B = 0$ but $A \neq 0$, it is a nonzero constant. Thus neither nonzero case can cancel the exponential in its growing direction.

When $r > 0$, that direction is $t \rightarrow \infty$; when $r < 0$, it is $t \rightarrow -\infty$. Hence the complete two-sided classification is

$$\boxed{r \neq 0 : y = 0 \text{ only}}, \quad \boxed{r = 0 : y = A \text{ for any constant } A}.$$

Step 4: Explain the exceptional repeated zero. A single zero root paired with a negative root gives a constant mode plus a decaying exponential, so every solution is forward bounded. A repeated zero root additionally gives the unbounded mode t .

For example, $y(t) = t$ solves $y'' = 0$, whose roots are both zero and therefore nonpositive, but is unbounded forward. Root signs alone are insufficient at a repeated zero; the polynomial multiplier must be included. For a strictly negative repeated root, the polynomial affects the rate and shape but not eventual decay.

Question 6

The repeated-root response $y(t) = te^{-t}$ solves $y'' + 2y' + y = 0$ with $y(0) = 0$, $y'(0) = 1$. Consider exponential bounds on $t \geq 0$.

Tasks

1. Find the maximum of y and verify that it tends to zero.
2. Determine whether any finite constant C can satisfy $y(t) \leq Ce^{-t}$ for every $t \geq 0$.
3. For a fixed $0 < \varepsilon < 1$, find the smallest C_ε such that $y(t) \leq C_\varepsilon e^{-(1-\varepsilon)t}$ for every $t \geq 0$. Prove optimality and sketch the bound for $\varepsilon = 1/2$ with its contact point.
4. Extend the argument to $w(t) = (A + Bt)e^{-t}$ by giving an explicit finite bound with rate $1 - \varepsilon$. Explain why eventual decay does not automatically give a bound with the exact exponent 1.

Question 6 – Solution

Strategy. Divide by the proposed exponential bound and maximize the resulting ratio, rather than judging only the root's sign.

Step 1: Find the transient peak. We have $y' = (1 - t)e^{-t}$, so the response rises until $t = 1$ and falls thereafter. Hence

$$\max_{t \geq 0} y(t) = 1/e.$$

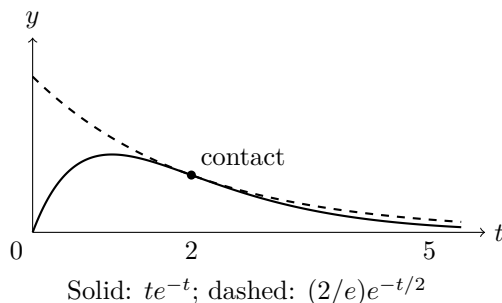
Also $t/e^t \rightarrow 0$, for example by l'Hôpital's rule, proving eventual decay.

Step 2: Test the exact root exponent. The proposed bound is equivalent to $t \leq C$ for all $t \geq 0$, which no finite C can satisfy. The exponential decay is modified by an unbounded polynomial prefactor.

Step 3: Find the sharp slightly slower bound. Dividing by $e^{-(1-\varepsilon)t}$ gives the ratio $te^{-\varepsilon t}$. Its derivative is $e^{-\varepsilon t}(1 - \varepsilon t)$, so its unique maximum occurs at $t = 1/\varepsilon$ and equals $1/(e\varepsilon)$. Thus

$$C_\varepsilon = \frac{1}{e\varepsilon}.$$

This constant is sufficient by the maximum calculation and necessary because equality holds at that time.



For $\varepsilon = 1/2$, the bound is $(2/e)e^{-t/2}$ and contact occurs at $t = 2$, where both values are $2e^{-2}$. Both derivatives equal $-e^{-2}$, confirming tangency rather than a crossing.

Step 4: Bound every fixed repeated-root solution. The triangle inequality gives

$$|w(t)|e^{(1-\varepsilon)t} \leq |A|e^{-\varepsilon t} + |B|te^{-\varepsilon t} \leq |A| + \frac{|B|}{e\varepsilon}.$$

Therefore $|w(t)| \leq (|A| + |B|/(e\varepsilon))e^{-(1-\varepsilon)t}$. This displayed constant need not be optimal for arbitrary A, B . A bound with the exact exponent 1 exists precisely when $B = 0$; for $B \neq 0$, the ratio $|A + Bt|$ is unbounded even though w itself decays.

Question 7

Consider the parameter family

$$y'' + 2y' + (1 - \mu)y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad -1 < \mu < 1.$$

For convenience, $\sinh z = (e^z - e^{-z})/2$. You may use $\sin z/z \rightarrow 1$ and $\sinh z/z \rightarrow 1$ as $z \rightarrow 0$.

Tasks

1. Solve the IVP separately for $\mu > 0$, $\mu = 0$ and $\mu < 0$, identifying the root type in each case.
2. Prove that the formulas from both sides tend to the repeated-root solution for each fixed $t \geq 0$ as $\mu \rightarrow 0$.
3. Determine which of these solutions have positive-time zeros. For $\mu < 0$, find the first positive zero and its limit as $\mu \uparrow 0$.
4. Prove that every solution of every equation in the stated parameter range tends to zero as $t \rightarrow \infty$. Explain why convergence of the displayed IVP solutions on any fixed time window does not preserve their number of positive-time zeros.

Question 7 – Solution

Strategy. Center the characteristic equation at its repeated root and compare the resulting real, repeated and complex formulas without conflating their global zero patterns.

Step 1: Solve each root case. The characteristic polynomial is $(r+1)^2 - \mu$. If $\mu > 0$, set $s = \sqrt{\mu} \in (0, 1)$. The roots are $-1 \pm s$ and the data give

$$y_\mu = e^{-t} \frac{\sinh(st)}{s}.$$

At $\mu = 0$, the root is -1 twice, and the solution is $y_0 = te^{-t}$. If $\mu < 0$, set $b = \sqrt{-\mu} > 0$; the roots are $-1 \pm ib$, giving

$$y_\mu = e^{-t} \frac{\sin(bt)}{b}.$$

Each formula has value 0 and derivative 1 at zero and solves its stated equation.

Step 2: Recover the repeated-root limit from either side. For $t > 0$, factor the first formula as $te^{-t} \sinh(st)/(st)$ and the third as $te^{-t} \sin(bt)/(bt)$. The supplied limits yield $y_\mu(t) \rightarrow te^{-t}$ from either side. At $t = 0$, all values are zero directly. Thus the repeated-root solution is the common fixed-time limit, rather than a missing special value.

Step 3: Locate zeros before taking their limit. For $\mu \geq 0$, the displayed solution is strictly positive for every $t > 0$, since $t > 0$ and $\sinh(st) > 0$. For $\mu < 0$, its positive zeros are

$$t_n = \frac{n\pi}{\sqrt{-\mu}}, \quad n = 1, 2, \dots$$

In particular, $t_1 \rightarrow \infty$ as $\mu \uparrow 0$. The infinitely many oscillations move beyond every fixed finite observation window.

Step 4: Separate decay from zero counts. For $0 < \mu < 1$, both roots $-1 \pm \sqrt{\mu}$ are negative; at zero, both repeated-root modes decay; for $\mu < 0$, the exponential factor e^{-t} bounds every fixed sine/cosine combination. Hence every solution decays throughout the stated parameter range.

On any fixed $[0, T]$, the ratios in Step 2 converge uniformly because their arguments lie in intervals shrinking to zero and both ratios extend continuously with value 1 there. Yet for every negative μ there are infinitely many later zeros, while y_0 has none for $t > 0$. Convergence on fixed windows does not control zeros that escape to infinity.

Question 8

For the dimensionless equation

$$y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 1,$$

define the nonnegative quantity $E(t) = ((y'(t))^2 + (y(t))^2)/2$.

Tasks

1. Solve the IVP and find the time and height of the displacement maximum.
2. Differentiate E using the equation. Prove that E is strictly decreasing between any two distinct nonnegative times, even though its derivative can vanish at an isolated time.
3. Compute $E(0)$ and its value at the displacement maximum. Sketch y and E , and explain why increasing displacement is consistent with decreasing E .
4. Use the energy identity to calculate $\int_0^\infty (y')^2 dt$, and confirm the answer by integrating the explicit derivative.

Question 8 – Solution

Strategy. Distinguish a coordinate's amplitude from a quantity combining position and derivative, then integrate the exact dissipation identity.

Step 1: Solve and locate the transient maximum. The repeated root is -1 . The data give $y = (1 + 2t)e^{-t}$ and $y' = (1 - 2t)e^{-t}$. Hence the unique displacement maximum is

$$t_m = 1/2, \quad y(t_m) = 2e^{-1/2} > 1.$$

Step 2: Derive and interpret dissipation. Differentiating E and substituting $y'' = -2y' - y$ gives

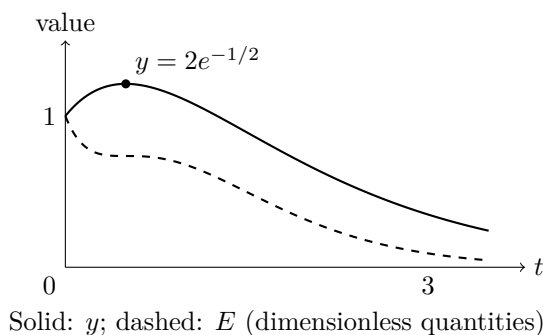
$$E' = y'y'' + yy' = y'(y'' + y) = -2(y')^2 \leq 0.$$

The derivative vanishes only at $t = 1/2$. On any interval $[s, t]$ with $s < t$, $(y')^2$ is positive except possibly at that one point, so its integral is strictly positive. Thus $E(t) - E(s) = -2 \int_s^t (y')^2 d\tau < 0$, proving strict decrease between distinct times.

Step 3: Compare the two quantities. Initially $E(0) = 1$. At the displacement peak, $y' = 0$, so

$$E(1/2) = \frac{1}{2}(2e^{-1/2})^2 = 2/e < 1.$$

The explicit expression is $E = (1 + 4t^2)e^{-2t}$.



Displacement initially increases, but the derivative contribution to E decreases enough for their sum to fall. The two curves represent the defined dimensionless quantities, not two competing claims about the same amplitude. Negative repeated roots ensure eventual decay without prohibiting an initial increase in y .

Step 4: Check the total integral in two ways. Since $y, y' \rightarrow 0$, also $E \rightarrow 0$. Integrating the dissipation identity gives $0 - 1 = -2 \int_0^\infty (y')^2 dt$, so the integral is $1/2$.

Directly, $(y')^2 = (1 - 4t + 4t^2)e^{-2t}$. Integration by parts gives the moments $\int_0^\infty e^{-2t} dt = 1/2$, $\int_0^\infty te^{-2t} dt = 1/4$ and $\int_0^\infty t^2 e^{-2t} dt = 1/4$. The resulting value $1/2 - 4(1/4) + 4(1/4) = 1/2$ agrees.

Question 9

For $L > 0$ and strictly positive endpoint values a, b , consider

$$y'' + 2y' + y = 0, \quad y(0) = a, \quad y(L) = b.$$

The same solution is defined beyond the observed interval.

Tasks

1. Find the unique solution and justify uniqueness for every $L > 0$.
2. Prove that it is strictly positive throughout $[0, L]$ using the transformed quantity $u = e^t y$.
3. Determine exactly which endpoint pairs make the response nonnegative for every future time $t \geq 0$, not just between observations.
4. For $L = 1$, $a = 1$, $b = e^{-2}$, find the first zero after the observed interval and determine the sign for all later times. Explain why positive endpoint observations did not guarantee permanent positivity.

Question 9 – Solution

Strategy. The exponential transformation turns the repeated-root endpoint problem into linear interpolation, but its continuation beyond the interval is extrapolation.

Step 1: Solve the endpoint equations. Write $y = (A + Bt)e^{-t}$. The first datum sets $A = a$, and the second gives $a + BL = be^L$. Thus

$$y(t) = \left[a + \frac{be^L - a}{L}t \right] e^{-t}.$$

The two constants are uniquely determined because $L \neq 0$. This also proves existence by substitution into the repeated-root family.

Step 2: Prove positivity between the data points. For $0 \leq t \leq L$,

$$u(t) = e^t y(t) = (1 - t/L)a + (t/L)be^L.$$

The weights are nonnegative and sum to 1, while both endpoint values are positive. Hence $u > 0$ and therefore $y > 0$ throughout the closed interval.

Step 3: Determine when positivity persists. The affine function $u = a + Bt$ starts positive. It remains nonnegative for all $t \geq 0$ exactly when $B \geq 0$. Equivalently,

$$b \geq ae^{-L}.$$

At equality, u is constant and $y = ae^{-t}$. If $b < ae^{-L}$, the slope of u is negative and its eventual zero is unavoidable despite both observed values being positive.

Step 4: Check the concrete continuation. Here $B = e^{-1} - 1 < 0$, so

$$y(t) = [1 - (1 - e^{-1})t]e^{-t}.$$

Its unique zero is

$$t_z = \frac{1}{1 - e^{-1}} = \frac{e}{e - 1} > 1.$$

The response is negative for every $t > t_z$ and tends to zero from below. The given endpoint at 1 is indeed $[e^{-1}]e^{-1} = e^{-2}$. Linear interpolation of positive transformed data stays positive between them; extending that same negative-slope line beyond the interval eventually crosses zero. The endpoint data determine the continuation uniquely, but do not force it to preserve the observed sign forever.

Question 10

A dimensionless family of repeated-root responses satisfies

$$y'' + 2\lambda y' + \lambda^2 y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad \lambda > 0.$$

The design requirement is $y(t) \leq 0.05$ for every $t \geq 2$, with the additional constraint $|y''(0)| \leq 4$.

Tasks

1. Solve the IVP and prove that the response is positive and strictly decreasing for $t > 0$.
2. Show that the first time at level 0.05 is z_*/λ , where $z_* > 0$ is the unique root of $(1+z)e^{-z} = 0.05$. Locate z_* between two consecutive four-decimal-place numbers by a justified numerical calculation.
3. Translate both the settling requirement and the initial-acceleration constraint into restrictions on λ . Determine whether they can be satisfied simultaneously.
4. Find the earliest achievable settling time under the acceleration constraint alone. Sketch the responses for $\lambda = 1, 2$ with the tolerance and required time marked, and explain why simply increasing λ is not an admissible unlimited improvement.

Question 10 – Solution

Strategy. Scale time by the repeated decay rate, then compare the performance requirement with the independent constraint on the initial acceleration.

Step 1: Solve and establish monotonicity. The repeated root is $-\lambda$. The data give

$$\boxed{y = (1 + \lambda t)e^{-\lambda t}}, \quad y' = -\lambda^2 t e^{-\lambda t}.$$

The response is positive, strictly decreasing for $t > 0$, starts at 1 and tends to zero. Therefore crossing the tolerance once is enough to guarantee it thereafter.

Step 2: Reduce the crossing calculation to one dimensionless number. Let $f(z) = (1 + z)e^{-z}$ for $z \geq 0$. Then $f(0) = 1$, $f(z) \rightarrow 0$ and $f'(z) = -ze^{-z} < 0$ for $z > 0$. The intermediate value theorem and strict decrease give exactly one positive root $f(z_*) = 0.05$.

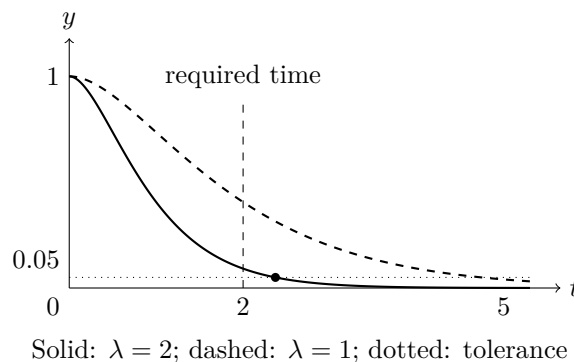
Direct evaluation gives $f(4.7438) > 0.05$ and $f(4.7439) < 0.05$. Thus

$$\boxed{4.7438 < z_* < 4.7439}, \quad t_{\text{settle}} = z_*/\lambda.$$

Numerically $z_* \approx 4.743865$; the bracket, rather than a rounded equality, supports the feasibility comparisons below.

Step 3: Test simultaneous feasibility. Settling by time 2 requires $z_*/\lambda \leq 2$, or $\lambda \geq z_*/2 > 2.3719$. But the equation at zero gives $y''(0) = -\lambda^2$, so the acceleration constraint requires $\lambda \leq 2$. These ranges do not intersect: the two requirements are incompatible.

At the largest admissible value $\lambda = 2$, the response at the deadline is $y(2) = 5e^{-4} \approx 0.091578 > 0.05$, confirming the failure directly.



Step 4: Optimize within the allowed family. The settling time z_*/λ strictly decreases as λ increases. Under $0 < \lambda \leq 2$, it is minimized at $\lambda = 2$, giving

$$\boxed{t_{\min} = z_*/2 \approx 2.371932}.$$

This is later than the required time 2. Larger λ would improve the settling time but violate the initial-acceleration bound. The conclusion optimizes this explicitly specified family and constraint, not every possible differential-equation model or control law.