

Question 1

Consider

$$2y'' + 4y' + 10y = 0, \quad y(1) = 2, \quad y'(1) = 2.$$

You may use Euler's identity $e^{i\theta} = \cos \theta + i \sin \theta$ and the continuous-coefficient linear initial-value uniqueness theorem.

Tasks

1. Find the characteristic roots. Use their complex exponential solutions to construct two real solutions.
2. Write a real solution family based at $t = 1$ and determine the coefficients from the initial data.
3. Verify the solution in the original equation and both data. Show that the real family can realize arbitrary initial value and slope at 1, and hence is complete.
4. In the complex form $c_+e^{(-1+2i)(t-1)} + c_-e^{(-1-2i)(t-1)}$, determine c_+, c_- for this IVP. Explain why taking both coefficients real and equal would lose some real solutions.

Question 1 – Solution

Strategy. Pair conjugate exponentials to obtain real functions, retaining two independent real data freedoms.

Step 1: Find the roots and real building blocks. The characteristic equation is $2r^2 + 4r + 10 = 0$, or $(r + 1)^2 + 4 = 0$, giving

$$r = -1 \pm 2i.$$

The half-sum and the difference divided by $2i$ of $e^{(-1 \pm 2i)t}$ give, respectively, $e^{-t} \cos 2t$ and $e^{-t} \sin 2t$. These solve the real equation because it is linear with real coefficients.

Step 2: Fit the shifted family. Set $x = t - 1$ and write $y = e^{-x}(A \cos 2x + B \sin 2x)$. At $x = 0$, $y = A$ and $y' = -A + 2B$. Thus $A = 2$ and $B = 2$, so

$$y = 2e^{-(t-1)}(\cos(2(t-1)) + \sin(2(t-1))).$$

Step 3: Verify the equation and completeness. Writing $y = e^{-x}u$ gives $y' = e^{-x}(u' - u)$ and $y'' = e^{-x}(u'' - 2u' + u)$. Hence

$$2y'' + 4y' + 10y = 2e^{-x}(u'' + 4u) = 0.$$

The value and slope at 1 are 2 and $-2 + 4 = 2$. More generally, data $y(1) = a$, $y'(1) = b$ give $A = a$, $B = (a + b)/2$, uniquely. Any solution shares its data with one family member; the linear theorem forces equality on $\mathbb{R}$.

Step 4: Interpret the complex constants correctly. The real combination satisfies

$$A \cos 2x + B \sin 2x = \frac{A - iB}{2} e^{2ix} + \frac{A + iB}{2} e^{-2ix}.$$

Thus $\boxed{c_+ = 1 - i, \quad c_- = 1 + i}$ for this IVP. The constants are conjugates, so the imaginary parts cancel. Equal real constants would force $B = 0$, permitting only initial slopes $b = -a$ in the shifted basis. Real solutions require conjugate coefficients, not necessarily real coefficients in the complex basis.

Question 2

Consider

$$y'' + 2y' + 5y = 0, \quad y(0) = -1, \quad y'(0) = 3.$$

Use the phase convention $y = Re^{-t} \cos(2t - \delta)$ with $R > 0$ and $0 \leq \delta < 2\pi$.

Tasks

1. Solve the IVP in sine/cosine form.
2. Determine R, δ in the stated convention and verify the initial value and derivative in phase form.
3. Find every zero for $t \geq 0$, the sign immediately after the first zero, and the spacing between successive zeros.
4. Explain why solving only $\tan \delta = B/A$ can give the wrong phase quadrant. Diagnose the choice $\delta = -\pi/4$ with positive amplitude for this IVP.

Question 2 – Solution

Strategy. Determine the sine and cosine coefficients before choosing the phase quadrant; then use the phase to locate zeros.

Step 1: Solve the real IVP. The roots are $-1 \pm 2i$. For $y = e^{-t}(A \cos 2t + B \sin 2t)$, the data give $A = -1$ and $-A + 2B = 3$. Thus $B = 1$ and

$$y = e^{-t}(-\cos 2t + \sin 2t).$$

Step 2: Choose the correct quadrant. Expansion of $R \cos(2t - \delta)$ gives $A = R \cos \delta$, $B = R \sin \delta$. Therefore

$$R = \sqrt{A^2 + B^2} = \sqrt{2}, \quad \cos \delta = -1/\sqrt{2}, \quad \sin \delta = 1/\sqrt{2}.$$

The prescribed convention selects $\delta = 3\pi/4$, yielding

$$y = \sqrt{2}e^{-t} \cos(2t - 3\pi/4).$$

At zero the value is -1 , while the derivative is $\sqrt{2}[-\cos(-3\pi/4) - 2\sin(-3\pi/4)] = 3$, as required.

Step 3: Enumerate the nonnegative zeros. The exponential is never zero. The first zero occurs when $2t - 3\pi/4 = -\pi/2$, so all nonnegative zeros are

$$t_n = \frac{\pi}{8} + \frac{n\pi}{2}, \quad n = 0, 1, 2, \dots$$

At t_0 , the derivative is $2\sqrt{2}e^{-\pi/8} > 0$, so the solution crosses from negative to positive. Successive zeros are separated by $\pi/2$, half the trigonometric period π . The decaying solution itself is not periodic.

Step 4: Resolve the tangent ambiguity. The equation $\tan \delta = -1$ does not distinguish quadrants II and IV. With positive $R = \sqrt{2}$, the phase $-\pi/4$ gives $A = 1$, $B = -1$, which is the negative of the required solution. It has value 1 and slope -3 at zero. The signs of both sine and cosine, together with the amplitude convention, determine the phase; a tangent value alone does not.

Question 3

For the response $y(t) = e^{-t} \cos 2t$ on $t \geq 0$, the curves $y = \pm e^{-t}$ are called its exponential envelopes because $|y(t)| \leq e^{-t}$.

Tasks

1. Find a monic homogeneous equation with this solution and verify its initial value and slope.
2. Find all positive-time stationary points, classify them, and identify the first minimum and the first positive-time maximum.
3. Determine where the response touches an envelope. Are these times stationary points? Find the ratio of each stationary-point magnitude to the envelope height there.
4. Sketch the response and envelopes on $[0, 2\pi]$, marking the first minimum. Determine the ratio of consecutive positive local maxima and explain why envelope contacts and extrema must be distinguished.

Question 3 – Solution

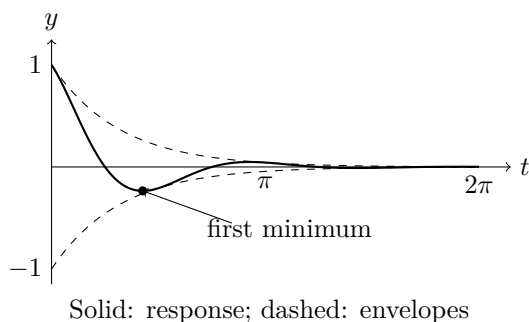
Strategy. Differentiate the entire product, including the exponential factor, before locating peaks.

Step 1: Verify the complex-root model. The roots $-1 \pm 2i$ give $y'' + 2y' + 5y = 0$. Differentiation yields $y' = e^{-t}(-\cos 2t - 2\sin 2t)$, so $y(0) = 1$ and $y'(0) = -1$. Substituting a second derivative verifies the equation.

Step 2: Locate and classify the extrema. Let $\theta = \arctan(1/2) \in (0, \pi/2)$. The stationary condition is $\cos 2t + 2\sin 2t = 0$, giving

$$t_k = (k\pi - \theta)/2, \quad k = 1, 2, \dots$$

At these times $\cos 2t_k = (-1)^k \cos \theta = (-1)^k 2/\sqrt{5}$. Since $y' = 0$, the ODE gives $y'' = -5y$. Thus odd k give negative minima and even k give positive maxima. The first minimum is at $(\pi - \theta)/2$; the first positive-time maximum is at $\pi - \theta/2$.



Step 3: Compare extrema with envelope contacts. Equality $|y| = e^{-t}$ requires $|\cos 2t| = 1$, or $t = n\pi/2$ for $n \geq 0$. At those times $y' = -y \neq 0$, so they are not stationary points. The response does share the tangent of the corresponding envelope there.

At every stationary point,

$$|y(t_k)| = \frac{2}{\sqrt{5}} e^{-t_k} < e^{-t_k}.$$

The decreasing envelope shifts the extrema away from cosine peaks.

Step 4: Compare like-signed peaks. Consecutive positive local maxima have indices k and $k + 2$, so their time separation is π . Their common trigonometric factor cancels, giving the height ratio $e^{-\pi}$. This is the envelope decay over a full cycle, even though the maxima occur strictly below the envelopes. At $t = 0$ the solution has a one-sided endpoint maximum; it is not a positive-time stationary maximum.

Question 4

A nonzero solution of an unknown monic equation $y'' + py' + qy = 0$ has characteristic roots $\alpha \pm i\beta$, where $\beta > 0$. Two consecutive positive local maxima occur at $t = 1$ and $t = 3$, with respective heights 4 and 1. The equation has real constant coefficients, and there is no forcing.

Tasks

1. Determine β and α from the peak times and heights. Justify why positive maxima are separated by a full cycle even when $\alpha \neq 0$.
2. Recover p, q and write the equation exactly.
3. Reconstruct the unique solution from the peak data at $t = 1$. Verify that the point at $t = 3$ is also a positive local maximum of height 1.
4. If the word consecutive were removed, would the two peak observations still uniquely determine the equation? Describe the resulting possibilities for β and explain the ambiguity.

Question 4 – Solution

Strategy. Use the interval between like-signed extrema for frequency and their height ratio for the real part of the roots.

Step 1: Extract frequency and decay. Write $y = Re^{\alpha t} \cos(\beta t - \delta)$. Stationarity requires $\alpha \cos(\beta t - \delta) - \beta \sin(\beta t - \delta) = 0$. Successive extrema have phase separation π and alternate sign, so consecutive positive maxima are separated by $2\pi/\beta$.

The observed separation is 2, giving $\beta = \pi$. The trigonometric factors at those maxima agree, so $1/4 = e^{2\alpha}$ and

$$\alpha = -\ln 2.$$

Step 2: Recover the real coefficients. Let $k = \ln 2 > 0$. The characteristic polynomial is $(r + k)^2 + \pi^2$, hence

$$y'' + 2ky' + (k^2 + \pi^2)y = 0.$$

Step 3: Reconstruct from value and zero slope. Using $x = t - 1$, write $y = e^{-kx}(A \cos \pi x + B \sin \pi x)$. The peak gives $A = 4$ and $-kA + \pi B = 0$, so

$$y = e^{-k(t-1)} \left[4 \cos(\pi(t-1)) + \frac{4k}{\pi} \sin(\pi(t-1)) \right].$$

At $t = 3$, $x = 2$, and $y = 4e^{-2k} = 1$, $y' = 0$. At either point the equation gives $y'' = -(k^2 + \pi^2)y < 0$, verifying strict positive local maxima. The solution is unique because its value and slope at 1 are specified for a regular linear equation.

Step 4: Identify what consecutiveness contributes. Without that condition, the time interval 2 could contain m full cycles, where m is any positive integer. Then $\beta = m\pi$ while the height ratio still fixes $\alpha = -\ln 2$. Each choice gives a different equation and a solution with the stated peaks, but for $m > 1$ there are additional positive maxima between them. The observed heights alone do not resolve the missing cycle count.

Question 5

For a real parameter a , consider

$$y'' + 2ay' + (a^2 + 9)y = 0.$$

All qualitative claims concern $t \geq 0$. A solution is nontrivial if it is not identically zero.

Tasks

1. Find the roots and the real general solution for every a .
2. Classify when every solution tends to zero and when every solution is bounded. Describe the nontrivial behavior at $a = 0$.
3. Prove that every nontrivial solution has infinitely many simple zeros, with the same spacing for all a . Can any such solution be eventually of one strict sign?
4. For $a < 0$, distinguish unbounded magnitude from the assertion $|y(t)| \rightarrow \infty$. Give explicit sequences of times that prove the correct conclusions.

Question 5 – Solution

Strategy. Separate the exponential amplitude from the trigonometric zero pattern; growth of the former does not erase zeros of the latter.

Step 1: Find the universal complex-root form. The polynomial is $(r + a)^2 + 9$, so the roots are $-a \pm 3i$. Hence

$$y = e^{-at}(A \cos 3t + B \sin 3t).$$

For a nontrivial solution, $R = \sqrt{A^2 + B^2} > 0$ and some phase δ give $y = Re^{-at} \cos(3t - \delta)$.

Step 2: Classify decay and boundedness. If $a > 0$, $|y| \leq Re^{-at} \rightarrow 0$. At $a = 0$, all solutions are bounded, and every nontrivial one is periodic with least positive period $2\pi/3$ and does not tend to zero. If $a < 0$, the magnitudes at trigonometric peaks grow without bound. Therefore

$$\boxed{\text{all solutions decay} \iff a > 0}, \quad \boxed{\text{all solutions are bounded} \iff a \geq 0}.$$

The zero solution is bounded and decays for every parameter value.

Step 3: Track the zeros and their slopes. Zeros occur when $3t - \delta = \pi/2 + n\pi$, so

$$t_n = \frac{\delta + \pi/2 + n\pi}{3},$$

with all integers n giving nonnegative times retained. There are infinitely many, spaced by $\boxed{\pi/3}$. At each zero,

$$y'(t_n) = -3Re^{-at_n} \sin(\pi/2 + n\pi) \neq 0.$$

Thus every zero is simple and the sign changes there. No nontrivial solution is eventually strictly positive or strictly negative, even when its amplitude decays.

Step 4: State growth without an incorrect limit. For $a < 0$, choose large integers n and set $s_n = (\delta + 2n\pi)/3$. Then $y(s_n) = Re^{-as_n} \rightarrow +\infty$. At $u_n = (\delta + (2n+1)\pi)/3$, $y(u_n) = -Re^{-au_n} \rightarrow -\infty$. But along the zero sequence, $|y(t_n)| = 0$.

Consequently the solution has unbounded magnitude, $\limsup y = +\infty$ and $\liminf y = -\infty$, while $\liminf |y| = 0$. The limit $|y(t)| \rightarrow \infty$ is false. A growing envelope describes arbitrarily large excursions, not a lower bound on the response at every late time.

Question 6

For constants $a \in \mathbb{R}$, $b > 0$, consider

$$y'' + 2ay' + (a^2 + b^2)y = 0.$$

Define transformed coordinates

$$u = e^{at}y, \quad v = \frac{e^{at}(y' + ay)}{b}.$$

The axes in a plot of (u, v) are these transformed quantities, not time and displacement.

Tasks

1. Differentiate u, v and prove $u' = bv$, $v' = -bu$.
2. Deduce a conserved quantity and express its value using $y(0) = Y$, $y'(0) = V$. Relate it to the amplitude of the trigonometric factor in y .
3. For $a = 1$, $b = 2$, $Y = 1$, $V = -1$, find y, u, v . Sketch the transformed curve with its initial point, direction and quarter-cycle points.
4. Explain why the circle in these transformed coordinates does not imply constant amplitude of the original response. Determine exactly which original initial data make the conserved quantity zero and what solution follows.

Question 6 – Solution

Strategy. Remove the exponential factor and scale the remaining derivative so the two trigonometric coordinates have equal amplitudes.

Step 1: Derive the transformed equations. Directly, $u' = e^{at}(y' + ay) = bv$. Also,

$$v' = \frac{e^{at}}{b}(y'' + 2ay' + a^2y) = \frac{e^{at}}{b}(-b^2y) = -bu.$$

The original equation is essential in the second equality.

Step 2: Identify the invariant and amplitude. Differentiating gives $(u^2 + v^2)' = 2u(bv) + 2v(-bu) = 0$. At zero,

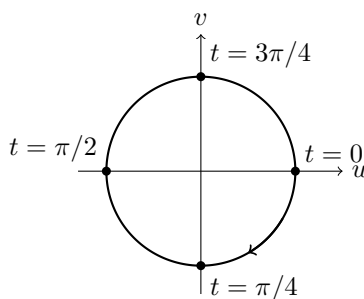
$$u^2 + v^2 = Y^2 + \frac{(V + aY)^2}{b^2} = R^2.$$

The real solution has trigonometric coefficients $A = Y$, $B = (V + aY)/b$, so $R = \sqrt{A^2 + B^2}$ is exactly its trigonometric amplitude.

Step 3: Trace the concrete circle. For the stated data, $A = 1$, $B = 0$. Thus

$$y = e^{-t} \cos 2t, \quad u = \cos 2t, \quad v = -\sin 2t.$$

The initial point is $(1, 0)$ and its velocity is $(0, -2)$, so motion is clockwise. At times $0, \pi/4, \pi/2, 3\pi/4$ the points are $(1, 0), (0, -1), (-1, 0), (0, 1)$, respectively; the cycle closes at $t = \pi$.



$$u^2 + v^2 = 1; \text{ equal axis scales}$$

Step 4: Translate back to the original quantity. The transformation removed e^{-at} : the original response satisfies $|y| \leq Re^{-at}$. Its amplitude decays for $a > 0$, is constant for $a = 0$, and grows for $a < 0$. A circle in (u, v) describes a normalized oscillation, not constant original displacement amplitude or a claim about physical energy.

Because $b > 0$, the sum of squares R^2 is zero exactly when $Y = 0$ and $V + aY = 0$, hence exactly when $Y = V = 0$. These data give $y \equiv 0$, consistent with initial-value uniqueness.

Question 7

An unknown frequency $\beta > 0$ appears in

$$y'' + \beta^2 y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Exact observations at all nonnegative integer times report $y(n) = 0$.

Tasks

1. Solve the IVP in terms of β and determine every frequency compatible with the integer observations.
2. Explain why the observations, even with the known initial slope, do not imply the zero solution and do not identify a unique frequency.
3. An additional exact observation gives $y(1/2) = 1/\pi$. Determine which of the previously possible frequencies remain. Prove that no larger candidate frequency can fit it.
4. Sketch the candidates with $\beta = \pi$ and $\beta = 3\pi$ on $[0, 2]$, marking the integer sampling times. Explain what feature of the observation schedule hides their different oscillations.

Question 7 – Solution

Strategy. Solve before interpreting the data; a sampling grid can repeatedly land on zeros of a nonzero oscillation.

Step 1: Determine all admissible frequencies. The roots are $\pm i\beta$, and the data give

$$y(t) = \frac{\sin \beta t}{\beta}.$$

In particular, $y(1) = 0$ implies $\sin \beta = 0$, so $\beta = m\pi$ for a positive integer m . Conversely, each such frequency makes every integer observation zero. The complete candidate set is therefore

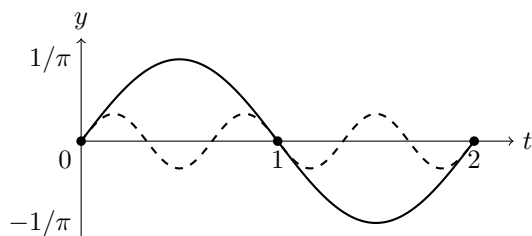
$$\beta = m\pi, \quad m = 1, 2, 3, \dots$$

Step 2: Retain the slope information correctly. Each candidate has $y'(t) = \cos(m\pi t)$ and hence $y'(0) = 1$, so each is nonzero. The zero function would violate that slope. The slope sets the sine coefficient to $1/(m\pi)$, but still allows infinitely many frequencies. A known initial slope does not rescue a schedule that misses all intermediate oscillations.

Step 3: Use the extra observation to select a frequency. For the candidate indexed by m ,

$$y(1/2) = \frac{\sin(m\pi/2)}{m\pi}.$$

For $m = 1$ this equals $1/\pi$. For every $m > 1$, its absolute value is at most $1/(m\pi) < 1/\pi$, making agreement impossible. Thus the new datum selects exactly $\beta = \pi$ and $y = \sin(\pi t)/\pi$.



Solid: $\beta = \pi$; dashed: $\beta = 3\pi$; dots: samples

Step 4: Explain the indistinguishable samples. The integer grid samples an integer number of half-cycles for every candidate, so it always observes a zero. The two plotted curves differ between those times, including in amplitude, yet share all grid values and the initial slope. This is an explicit sampling ambiguity, not a violation of uniqueness: each fixed equation has one solution for its data, while different frequencies define different equations.

Question 8

Consider the endpoint problem

$$y'' + 2y' + 5y = 0, \quad y(0) = A, \quad y(L) = B, \quad L > 0.$$

The numbers A, B are endpoint values, not mode coefficients.

Tasks

1. Derive the endpoint equation for the remaining sine coefficient. Determine all lengths L for which every pair of endpoint values gives a unique solution.
2. At each excluded length, give the exact compatibility condition for existence and classify the number of solutions.
3. For $A = 0, B = 1$, solve when $L = \pi/4$ and determine what happens at $L = \pi/2$.
4. Keep $A = 0, B = 1$ but take $L = \pi/2 + \varepsilon$, where $0 < \varepsilon < \pi/4$. Find $y(\pi/4)$ and its leading behavior as $\varepsilon \downarrow 0$. Explain how unique solvability for every such ε can coexist with arbitrarily large interior values.

Question 8 – Solution

Strategy. An exponential prefactor never vanishes, so solvability and sensitivity are governed by the sine factor in the endpoint data map.

Step 1: Reduce to one coefficient. The roots are $-1 \pm 2i$. The first endpoint fixes the cosine coefficient, giving $y = e^{-t}(A \cos 2t + C \sin 2t)$. The second requires

$$C \sin 2L = Be^L - A \cos 2L.$$

If $\sin 2L \neq 0$, there is exactly one C , so unique solvability for arbitrary data holds exactly when

$$L \neq n\pi/2 \quad (n = 1, 2, 3, \dots).$$

Step 2: Classify the exceptional lengths. At $L = n\pi/2$, the condition becomes $0 = Be^L - (-1)^n A$. Thus

$$B = (-1)^n e^{-L} A$$

is necessary and sufficient for existence. If it holds, every real C works, giving infinitely many solutions. If it fails, there is no solution. The initial-value uniqueness theorem does not promise endpoint uniqueness at these lengths.

Step 3: Apply the two specified lengths. For $A = 0$, $B = 1$, $L = \pi/4$, we have $C = e^{\pi/4}$ and

$$y = e^{\pi/4-t} \sin 2t.$$

At $L = \pi/2$, compatibility would require $B = 0$, contradicting $B = 1$, so no solution exists.

Step 4: Quantify sensitivity near an exceptional length. For $L = \pi/2 + \varepsilon$, $\sin 2L = -\sin 2\varepsilon \neq 0$. Hence

$$y(t) = -\frac{e^{L-t} \sin 2t}{\sin 2\varepsilon}, \quad y(\pi/4) = -\frac{e^{\pi/4+\varepsilon}}{\sin 2\varepsilon}.$$

Using $\sin 2\varepsilon \sim 2\varepsilon$ and $e^\varepsilon \rightarrow 1$ gives

$$y(\pi/4) \sim -\frac{e^{\pi/4}}{2\varepsilon} \longrightarrow -\infty.$$

For each positive ε there is one finite coefficient, but the denominator approaches zero as the length approaches the incompatible endpoint problem. Uniqueness at each parameter value is not a uniform bound on the dependence on that parameter.

Question 9

For the decaying oscillation $y(t) = e^{-t} \sin t$, define its signed total area and its total absolute area by

$$I = \int_0^\infty y(t) dt, \quad J = \int_0^\infty |y(t)| dt.$$

Tasks

1. Find the corresponding monic homogeneous equation and initial data. State the signs on consecutive half-cycles.
2. Find an antiderivative of $e^{-t} \sin t$ and calculate I .
3. Calculate J exactly by comparing the absolute areas of successive half-cycles. Explain why $J > I$ even though both are finite.
4. Sketch the first two half-cycles with their signed areas indicated. Bound the absolute tail after an arbitrary $T \geq 0$, and give its exact value when $T = n\pi$ for a nonnegative integer n .

Question 9 – Solution

Strategy. Separate cancellation between lobes from decay of each lobe, then sum the absolute areas as a geometric series.

Step 1: Identify the equation and signs. The roots are $-1 \pm i$, so $y'' + 2y' + 2y = 0$. The derivative is $e^{-t}(\cos t - \sin t)$, giving $y(0) = 0$, $y'(0) = 1$. On $(n\pi, (n+1)\pi)$, the sign is $(-1)^n$: positive first, negative next, and alternating thereafter.

Step 2: Integrate the signed response. Integration by parts twice, or differentiation of the result, gives

$$F(t) = -\frac{1}{2}e^{-t}(\sin t + \cos t), \quad F' = e^{-t} \sin t.$$

Since $F(t) \rightarrow 0$ as $t \rightarrow \infty$ and $F(0) = -1/2$,

$$I = 1/2.$$

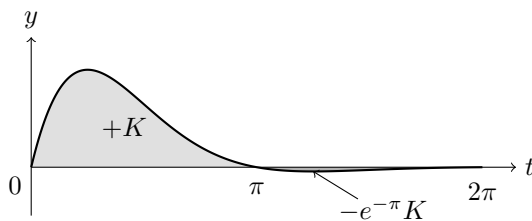
Step 3: Sum the magnitudes without cancellation. The first positive lobe has area

$$K = F(\pi) - F(0) = \frac{1 + e^{-\pi}}{2}.$$

The substitution $t = n\pi + s$ shows that the absolute area of the n th lobe is $e^{-n\pi}K$. Thus the convergent geometric sum gives

$$J = \frac{K}{1 - e^{-\pi}} = \frac{1 + e^{-\pi}}{2(1 - e^{-\pi})} > \frac{1}{2}.$$

The negative lobes reduce I but contribute positively to J . Infinitely many oscillations do not prevent absolute convergence because the lobe magnitudes decrease geometrically.



Signed lobe areas; both magnitudes contribute to J .

Step 4: Control the unobserved tail. Since $|\sin t| \leq 1$,

$$\int_T^\infty |y(t)| dt \leq \int_T^\infty e^{-t} dt = e^{-T}.$$

At a half-cycle boundary, substitution by $n\pi$ preserves $|\sin|$, so the exact absolute tail is $e^{-n\pi}J$. The signed area of the second plotted lobe is $-e^{-\pi}K$, while its contribution to J is $+e^{-\pi}K$.

Question 10

Consider

$$y'' + 4y' + 13y = 0, \quad y(0) = 3, \quad y'(0) = 6.$$

A settling certificate at time $T \geq 0$ means a proof that $|y(t)| \leq 0.1$ for every $t \geq T$. Distinguish a certificate for this specific solution from one valid for every phase with the same trigonometric amplitude.

Tasks

1. Find the solution, its amplitude R , and its phase $\delta \in [0, 2\pi)$ in the form $Re^{-2t} \cos(3t - \delta)$.
2. Use the envelope to find the smallest certificate time valid for every phase with amplitude R . Prove the claimed optimality over phases.
3. For the stated IVP, show that $T = 1.9$ is already a certificate. Use the value at 1.9 and the magnitudes of all later stationary points, justifying their ordering. Numerical evaluations should be supported by exact expressions.
4. Sketch the response for $t \geq 1.5$ with the tolerance levels and envelope certificate marked. Explain why the envelope certificate need not be the earliest settling time for one fixed phase, and whether Task 3 identifies that exact earliest time.

Question 10 – Solution

Strategy. First obtain a phase-independent bound, then sharpen it for one solution by controlling every later extremum.

Step 1: Resolve the data and phase. The roots are $-2 \pm 3i$. Writing $y = e^{-2t}(A \cos 3t + B \sin 3t)$ gives $A = 3$ and $-2A + 3B = 6$, hence $B = 4$. Thus

$$y = e^{-2t}(3 \cos 3t + 4 \sin 3t) = 5e^{-2t} \cos(3t - \delta), \quad \delta = \arctan(4/3) \in (0, \pi/2).$$

Step 2: Obtain the best certificate for every phase. The envelope condition is $5e^{-2t} \leq 0.1$, so

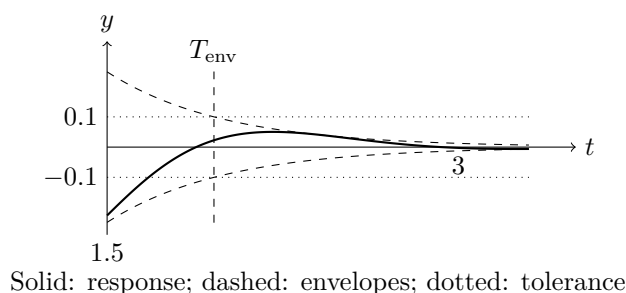
$$T_{\text{env}} = \frac{1}{2} \ln 50 \approx 1.956012.$$

It guarantees the tolerance at every later time. For any proposed $0 \leq T < T_{\text{env}}$, choose a phase with $3T - \delta$ an integer multiple of 2π . That solution has $|y(T)| = 5e^{-2T} > 0.1$. Thus no earlier time works for every phase with amplitude 5.

Step 3: Check the fixed phase more sharply. Let $\theta = \arctan(2/3)$. Stationary times are $t_k = (\delta - \theta + k\pi)/3$, and their magnitudes are $(15/\sqrt{13})e^{-2t_k}$. In particular,

$$t_1 \approx 1.160295 < 1.9 < t_2 \approx 2.207493, \quad |y(t_2)| \approx 0.050317 < 0.1.$$

Also $y(1.9) = e^{-3.8}(3 \cos 5.7 + 4 \sin 5.7) \approx 0.006742$. Between 1.9 and t_2 there is no stationary point, so the response is monotone and its absolute value is bounded by the larger endpoint magnitude. Each subsequent stationary magnitude is multiplied by $e^{-2\pi/3} < 1$. Monotonicity between those extrema then proves $|y(t)| \leq 0.1$ for all $t \geq 1.9$.



Step 4: Distinguish a certificate from the exact last crossing. The envelope treats all phases, whereas the specified phase can have small values while the envelope is still above 0.1. The preceding negative minimum has magnitude greater than 0.1, so a final tolerance crossing occurs between t_1 and 1.9. Task 3 proves a sufficient time, not that 1.9 is the earliest one; locating the exact last crossing requires solving the corresponding trigonometric-exponential level equation.