

Question 1

Consider

$$6y'' + y' - 2y = 0, \quad y(1) = 7, \quad y'(1) = 0.$$

For a constant-coefficient homogeneous second-order equation with distinct real characteristic roots r_1, r_2 , the general solution is $C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

Tasks

1. Substitute $y = e^{rt}$ to derive the characteristic equation. Find and verify both roots.
2. Write the general solution using exponentials based at $t = 1$, then determine its constants from the initial data.
3. Verify the resulting function in the original equation and both initial conditions. Explain why it is the unique solution on $\mathbb{R}$.
4. Classify the stationary point at $t = 1$ and determine the limits as $t \rightarrow \infty$ and $t \rightarrow -\infty$. Explain why zero initial slope does not mean a constant solution.

Question 1 – Solution

Strategy. Keep the leading coefficient in the characteristic polynomial and choose a shifted exponential basis to simplify the data.

Step 1: Derive, factor and check the polynomial. Substitution gives $(6r^2 + r - 2)e^{rt} = 0$. Since $e^{rt} \neq 0$,

$$6r^2 + r - 2 = (2r - 1)(3r + 2) = 0, \quad \boxed{r_1 = 1/2, \quad r_2 = -2/3}.$$

Direct evaluation gives $6(1/2)^2 + 1/2 - 2 = 0$ and $6(-2/3)^2 - 2/3 - 2 = 0$.

Step 2: Use the actual initial time. A shifted basis only rescales the arbitrary constants, so write

$$y = Ae^{(t-1)/2} + Be^{-2(t-1)/3}.$$

The data give $A + B = 7$ and $A/2 - 2B/3 = 0$. Thus $A = 4$, $B = 3$, and

$$\boxed{y = 4e^{(t-1)/2} + 3e^{-2(t-1)/3}}.$$

Step 3: Verify the equation and uniqueness. Each exponential contributes its coefficient times $6r^2 + r - 2$ to the residual, so their sum solves the original equation. At 1, its value is $4 + 3 = 7$ and its derivative is $2 - 2 = 0$. After division by 6, all coefficients are continuous on $\mathbb{R}$. The linear initial-value theorem therefore gives uniqueness there.

Step 4: Interpret the initial balance of modes. Differentiating twice gives

$$y'' = e^{(t-1)/2} + \frac{4}{3}e^{-2(t-1)/3} > 0.$$

Thus y' is strictly increasing, crosses zero at 1, and $\boxed{y(1) = 7}$ is the unique global minimum. The positive-root mode gives $y \rightarrow +\infty$ as $t \rightarrow \infty$; the negative-root mode gives $y \rightarrow +\infty$ as $t \rightarrow -\infty$.

Zero slope at one instant is a balance between two changing modes. Indeed the equation at 1 gives $y''(1) = 7/3$, not zero. A constant solution would have to satisfy $-2y = 0$ and could not have value 7.

Question 2

Consider the family of initial-value problems

$$y'' - y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = v.$$

Call a solution forward bounded if $|y(t)|$ is bounded for $t \geq 0$.

Tasks

1. Find the roots and express the solution coefficients in terms of v .
2. Determine exactly which initial slope produces a forward-bounded solution. State its limit.
3. Replace that slope by $v = -2 + \varepsilon$, where $0 < \varepsilon < 1$. Find when the growing and decaying terms have equal magnitude, and determine the eventual behavior.
4. For $\varepsilon = 0.01$, calculate that time to three decimal places and sketch the perturbed and unperturbed solutions on $[0, 2]$. Explain why a small initial error can change the long-time conclusion.

Question 2 – Solution

Strategy. Express the initial data in the two modes and isolate the coefficient of the growing exponential.

Step 1: Resolve the data into modes. The characteristic polynomial is $(r - 3)(r + 2)$, so $y = Ae^{3t} + Be^{-2t}$. From $A + B = 1$ and $3A - 2B = v$,

$$A = (v + 2)/5, \quad B = (3 - v)/5.$$

The nonzero root separation makes this data system invertible.

Step 2: Remove the growing mode exactly. If $A \neq 0$, the term Ae^{3t} dominates and $|y| \rightarrow \infty$. Thus forward boundedness holds exactly when $A = 0$, or $v = -2$. The selected solution is $y = e^{-2t}$ and tends to zero.

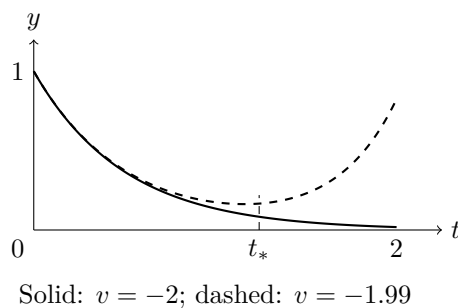
Step 3: Track a small slope perturbation. For $v = -2 + \varepsilon$,

$$y_\varepsilon = \frac{\varepsilon}{5}e^{3t} + \frac{5 - \varepsilon}{5}e^{-2t}.$$

Both terms are positive. They are equal precisely when $e^{5t} = (5 - \varepsilon)/\varepsilon$, giving

$$t_* = \frac{1}{5} \ln \frac{5 - \varepsilon}{\varepsilon}.$$

This is positive for the stated range of ε . Eventually $y_\varepsilon \rightarrow +\infty$, although its initial slope is negative.



Step 4: Quantify the delayed disagreement. For $\varepsilon = 0.01$, $t_* = (\ln 499)/5 \approx 1.243$. The exact difference is

$$y_\varepsilon - e^{-2t} = \frac{\varepsilon}{5}(e^{3t} - e^{-2t}).$$

It vanishes initially and is small with ε on any fixed finite interval, but grows without bound as $t \rightarrow \infty$ for every fixed positive ε . Forward boundedness here requires exact cancellation of a mode; it is not robust to a nonzero slope error. The time t_* compares mode magnitudes and is not the time of the trajectory's minimum.

Question 3

A transient response satisfies

$$y'' + 5y' + 6y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Only $t \geq 0$ is considered in the qualitative questions.

Tasks

1. Solve the initial-value problem using its real distinct roots.
2. Prove that $y(t) > 0$ for $t > 0$, even though $y(t) \rightarrow 0$ as $t \rightarrow \infty$.
3. Find the unique maximum, including its time and height, and sketch the response with that point marked.
4. Calculate $\int_0^\infty y(t) dt$ in two ways: from the explicit formula and by integrating the differential equation. Explain why negative roots alone do not imply monotone decay from the initial value.

Question 3 – Solution

Strategy. Separate eventual decay from the shape created by two modes with opposite coefficients.

Step 1: Solve for the two coefficients. The characteristic equation is $(r + 2)(r + 3) = 0$, so $y = Ae^{-2t} + Be^{-3t}$. The data give $A + B = 0$ and $-2A - 3B = 1$. Hence

$$y = e^{-2t} - e^{-3t}.$$

Its value and derivative at zero are 0 and 1, and both exponential terms satisfy the equation.

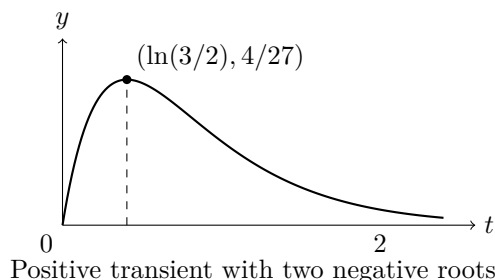
Step 2: Prove positivity and decay. For $t > 0$, $y = e^{-3t}(e^t - 1) > 0$. Both exponential terms tend to zero, so the positive response eventually decays to zero. Positivity does not require both mode coefficients to be positive.

Step 3: Locate and evaluate the only peak. The derivative factors as

$$y' = e^{-3t}(3 - 2e^t).$$

It is positive before $t_m = \ln(3/2)$ and negative afterward. At that time $e^{-t_m} = 2/3$, so

$$t_m = \ln(3/2), \quad y(t_m) = 4/9 - 8/27 = 4/27.$$



Step 4: Check the total area independently. Direct integration gives $\int_0^\infty y \, dt = 1/2 - 1/3 = \boxed{1/6}$. Alternatively, integrate the equation on $[0, T]$:

$$y'(T) - y'(0) + 5(y(T) - y(0)) + 6 \int_0^T y \, dt = 0.$$

The explicit formula verifies $y(T), y'(T) \rightarrow 0$ and convergence of the integral. Passing to the limit gives $-1 + 6 \int_0^\infty y \, dt = 0$, confirming the result.

Negative roots control the eventual limit of each mode. Their combination can initially increase: this solution starts at zero with positive slope, reaches a peak, and only then decreases.

Question 4

For the equation

$$y'' + 3y' + 2y = 0, \quad y(0) = a > 0, \quad y'(0) = b,$$

classify the entire range of real initial slopes b on the time interval $t \geq 0$.

Tasks

1. Find the solution in terms of a, b .
2. Determine exactly when $y(t) \geq 0$ for every $t \geq 0$. Prove both necessity and sufficiency.
3. Determine exactly when the solution is both nonnegative and nonincreasing on $[0, \infty)$, including boundary slopes.
4. For the remaining slopes, find the unique zero when one occurs, or the unique positive-time maximum when one occurs. Summarize the cases on a diagram of the ratio b/a .

Strategy. Factor out a positive exponential and reduce sign questions to affine expressions in $z = e^{-t} \in (0, 1]$.

$$y = (2a + b)e^{-t} - (a + b)e^{-2t}.$$

Step 2: Classify nonnegativity. Since $y = e^{-t}(A + Bz)$, its sign is the sign of an affine function with values A at $z = 0$ and $a > 0$ at $z = 1$. If $A \geq 0$, it is positive for all $0 < z \leq 1$. If $A < 0$, it is negative for sufficiently small positive z . Therefore

$$\boxed{y(t) \geq 0 \text{ for all } t \geq 0 \quad \Longleftrightarrow \quad b \geq -2a}$$

Step 3: Impose monotonicity as well. We have $y' = -e^{-t}(A + 2Bz)$. Its bracket has endpoint values A and $A + 2B = -b$. Thus it is nonnegative on $0 < z \leq 1$ exactly when $A \geq 0$ and $b \leq 0$. Combining the conditions yields

$$\boxed{-2a \leq b \leq 0}.$$

One zero; negative tail Positive and nonincreasing Positive; one peak

← -2 0 → b/a

—•————•—

Step 4: Resolve the two remaining regimes. If $b < -2a$, solving $A + Be^{-t} = 0$ gives the unique zero

$$t_z = \ln \frac{a+b}{2a+b} > 0.$$

If $b > 0$, it remains positive but initially rises. Solving $y' = 0$ gives the unique positive-time maximum at

$$t_m = \ln \frac{2(a+b)}{2a+b} > 0.$$

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Question 5

An unknown monic equation $y'' + py' + qy = 0$ has distinct real roots r_1, r_2 . A solution is known to contain both modes with nonzero coefficients. Its exact samples are

$$S_n = y(n \ln 2), \quad S_0 = 3, \quad S_1 = 1, \quad S_2 = 3/8, \quad S_3 = 5/32.$$

Set $z_j = e^{r_j \ln 2} > 0$, so $S_n = Az_1^n + Bz_2^n$.

Tasks

1. Derive a recurrence $S_{n+2} = sS_{n+1} - kS_n$ in terms of z_1, z_2 .
2. Use the four samples to determine s, k . Explain why this pair is uniquely determined.
3. Recover the real roots, the coefficients p, q , the solution $y(t)$ and its initial slope. Verify all four samples.
4. Explain what would fail if the observed solution contained only one mode. Could arbitrarily many exact samples of one pure exponential identify both roots of an unknown second-order equation?

Question 5 – Solution

Strategy. Recover the two exponential multipliers from a sample recurrence, then convert those multipliers back into differential-equation roots.

Step 1: Derive the recurrence. Each z_j satisfies $z_j^2 - (z_1 + z_2)z_j + z_1z_2 = 0$. Multiplying by its mode coefficient and z_j^n , then summing, gives

$$\boxed{S_{n+2} = sS_{n+1} - kS_n}, \quad s = z_1 + z_2, \quad k = z_1z_2.$$

Step 2: Solve the sample equations exactly. For $n = 0, 1$,

$$s - 3k = 3/8, \quad (3/8)s - k = 5/32.$$

Substitution of $s = 3/8 + 3k$ into the second equation gives $k/8 = 1/64$. Hence

$$\boxed{k = 1/8, \quad s = 3/4}.$$

The nonzero coefficient $1/8$ in this elimination proves uniqueness. Equivalently, $S_0S_2 - S_1^2 = 1/8 \neq 0$.

Step 3: Recover and verify the continuous model. The multipliers solve $z^2 - (3/4)z + 1/8 = (z - 1/2)(z - 1/4) = 0$. Since $r = \ln z / \ln 2$, the roots are $-1, -2$. Thus $p = 3, q = 2$.

Writing $y = Ae^{-t} + Be^{-2t}$, the first two samples give $A + B = 3, A/2 + B/4 = 1$, so $A = 1, B = 2$. Therefore

$$\boxed{y = e^{-t} + 2e^{-2t}}, \quad \boxed{y'(0) = -5}.$$

The four sample values are $3, 1/2 + 2/4 = 1, 1/4 + 2/16 = 3/8$, and $1/8 + 2/64 = 5/32$, as required. Both coefficients are nonzero.

Step 4: Identify the missing-information case. For a pure observed mode $S_n = Cz^n$, the recurrence only imposes $z^2 - sz + k = 0$. Any positive second multiplier different from z gives another pair s, k with exactly the same samples. Thus the active root can be identified, but an unexcited root remains undetermined, even with arbitrarily many noiseless samples of that one trajectory. Exact data do not reveal a mode whose coefficient is zero.

Question 6

Consider homogeneous constant-coefficient equations with two distinct real roots $r_1 < r_2$ and endpoint data $y(0) = a$, $y(L) = b$, where $L > 0$.

Tasks

1. Prove that every pair $a, b \in \mathbb{R}$ determines exactly one solution by solving for the two mode coefficients. Identify the quantity that cannot vanish.
2. Solve $y'' - 3y' + 2y = 0$ with $y(0) = y(\ln 2) = 1$.
3. Prove that a nonzero combination $Ae^{r_1 t} + Be^{r_2 t}$ has at most one real zero. Treat vanishing coefficients explicitly, and deduce positivity between strictly positive endpoints.
4. For the solution in Task 2, find its maximum on $[0, \ln 2]$ and sketch it. Does positivity between positive endpoints mean the solution stays below its endpoint values?

Question 6 – Solution

Strategy. Use the strict inequality of real exponentials to solve the endpoint problem and control the number of zeros.

Step 1: Invert the endpoint constraints. For $y = Ae^{r_1 t} + Be^{r_2 t}$, the conditions give $A + B = a$ and $Ae^{r_1 L} + Be^{r_2 L} = b$. Therefore

$$B = \frac{b - ae^{r_1 L}}{e^{r_2 L} - e^{r_1 L}}, \quad A = a - B.$$

Because $r_2 > r_1$ and $L > 0$, the denominator is strictly positive. There is exactly one coefficient pair for all endpoint data, unlike some oscillatory endpoint problems.

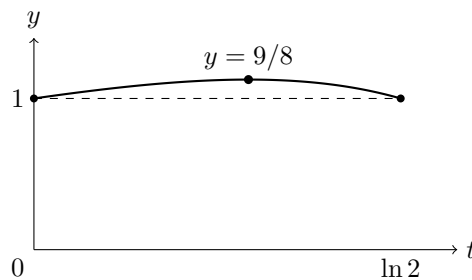
Step 2: Fit the concrete equation. Here the roots are 1 and 2. At $L = \ln 2$, the equations are $A + B = 1$ and $2A + 4B = 1$. Hence

$$y = \frac{3}{2}e^t - \frac{1}{2}e^{2t}.$$

Both endpoints have value 1, and each mode satisfies the original equation.

Step 3: Bound the number of zeros and deduce positivity. Factor the general solution as $e^{r_1 t}(A + Be^{(r_2 - r_1)t})$. If $B \neq 0$, the bracket is strictly monotone, so it has at most one zero. If $B = 0$, a nonzero solution is a nonvanishing pure exponential; the case $A = 0$ is likewise nonvanishing.

If a solution with positive endpoints became negative inside, continuity would force two distinct zeros. If it merely touched zero inside while staying nonnegative, that interior minimum would have $y = y' = 0$; initial-value uniqueness would force the identically zero solution, contradicting the endpoints. Thus it is strictly positive throughout the interval.



Positive endpoints do not bound the interior maximum.

Step 4: Distinguish positivity from an upper bound. For the concrete solution, $y' = e^t(3/2 - e^t)$ changes from positive to negative at $t = \ln(3/2)$, which lies between 0 and $\ln 2$. Its value there is

$$y_{\max} = \frac{3}{2} \frac{3}{2} - \frac{1}{2} \frac{9}{4} = \frac{9}{8} > 1.$$

The curve stays positive but exceeds both endpoint values. A restriction on zeros is not a maximum principle bounding the response by its endpoints.

Question 7

For a real parameter α , consider

$$y'' + 2\alpha y' + (\alpha^2 - 4)y = 0.$$

A solution is forward bounded if it is bounded on $[0, \infty)$. Distinguish statements about every solution from statements about selected nonzero solutions.

Tasks

1. Find both roots and show that they are real and distinct for every real α .
2. Determine exactly when every solution tends to zero as $t \rightarrow \infty$.
3. Determine exactly when every solution is forward bounded. Treat $\alpha = 2$ explicitly.
4. For all $\alpha \leq 2$, classify the individual forward-bounded solutions, including $\alpha = -2$. Give a complete parameter table and explain why a zero root must not be treated as a negative root.

Question 7 – Solution

Strategy. Track the two root signs and retain the distinction between a mode being available and its coefficient actually being present.

Step 1: Keep the root separation visible. The characteristic polynomial is $(r + \alpha)^2 - 4$, so

$$\boxed{r_+ = 2 - \alpha, \quad r_- = -2 - \alpha}, \quad r_+ - r_- = 4.$$

Consequently $y = Ae^{(2-\alpha)t} + Be^{(-2-\alpha)t}$ for every real α ; no repeated or nonreal roots occur.

Step 2: Require both modes to decay. Every solution tends to zero exactly when both roots are negative. Since r_+ is the larger root, this is equivalent to

$$\boxed{\alpha > 2}.$$

If $\alpha \leq 2$, choosing $A = 1$, $B = 0$ gives a constant or growing mode, disproving universal decay.

Step 3: Allow a constant mode for boundedness. Every solution is forward bounded exactly when both roots are nonpositive, or $\boxed{\alpha \geq 2}$. At $\alpha = 2$,

$$y = A + Be^{-4t} \rightarrow A.$$

All solutions are bounded, but only those with $A = 0$ tend to zero. A zero root supplies a constant mode rather than a decaying one.

Step 4: Classify the selected bounded solutions. Using the coefficients A, B from Step 1, the full classification is

α	forward-bounded solutions	their limits
$\alpha < -2$	$y = 0$	0
$\alpha = -2$	$y = B \quad (A = 0)$	B
$-2 < \alpha < 2$	$y = Be^{(-2-\alpha)t} \quad (A = 0)$	0
$\alpha = 2$	$y = A + Be^{-4t}$	A
$\alpha > 2$	all solutions	0

For $\alpha < -2$, both roots are positive. If $A \neq 0$, the faster mode forces unbounded magnitude; if $A = 0$ and $B \neq 0$, the remaining positive mode still grows. Distinct growth rates cannot cancel for all large times.

At $\alpha = -2$ the roots are 4 and 0, so only constant solutions are bounded. For $-2 < \alpha < 2$, one positive mode must be removed, leaving a decaying mode. These boundary cases show why a classification based only on whether roots are real would be incomplete.

Question 8

The positive response

$$y(t) = e^{-t} + 9e^{-3t} \quad (t \geq 0)$$

solves a homogeneous second-order equation with real distinct roots. Define its instantaneous decay rate by $R(t) = -y'(t)/y(t)$.

Tasks

1. Recover the monic differential equation and verify the initial value and slope.
2. Express R as a weighted average of the two decay rates. Find $R(0)$, its limit, and the unique time when $R = 2$.
3. Differentiate R using the differential equation to prove that it is strictly decreasing. Sketch R with its limiting level marked.
4. A constant-rate extrapolation from the initial data predicts $\hat{y}(t) = 10e^{-(14/5)t}$. Prove whether this estimate lies above or below the actual response for every $t > 0$, using the curvature of $\ln y$.

Question 8 – Solution

Strategy. A mixture of two exponential modes need not have a constant effective decay rate; use its logarithmic derivative to quantify the change.

Step 1: Recover and check the equation. The roots are $-1, -3$, so the monic characteristic polynomial is $r^2 + 4r + 3$. Hence $y'' + 4y' + 3y = 0$. Direct differentiation gives $y' = -e^{-t} - 27e^{-3t}$, so

$$y(0) = 10, \quad y'(0) = -28.$$

The response is positive because both mode terms are positive.

Step 2: Read the rate as a changing weighted average. Let $z = 9e^{-2t} > 0$. Then

$$R = \frac{e^{-t} + 27e^{-3t}}{e^{-t} + 9e^{-3t}} = \frac{1 + 3z}{1 + z}.$$

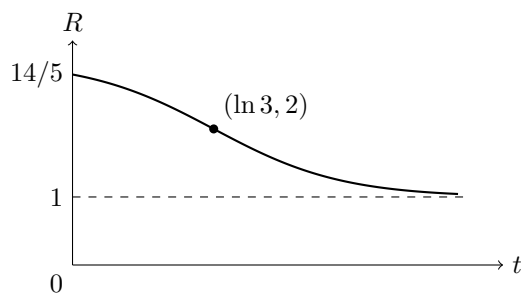
This averages 1 and 3 with positive weights 1, z , so $1 < R < 3$. In particular,

$$R(0) = 14/5, \quad R \rightarrow 1, \quad R = 2 \iff z = 1 \iff t = \ln 3.$$

Step 3: Derive the rate equation. Using $y'' = -4y' - 3y$ and $y'/y = -R$,

$$R' = -\frac{y''}{y} + (y'/y)^2 = R^2 - 4R + 3 = (R - 1)(R - 3) < 0.$$

Thus R is strictly decreasing; its long-time limit is the slower mode's rate, rather than an unchanging average of the roots.



The effective decay rate decreases toward 1.

Step 4: Compare with a constant-rate prediction. Set $f = \ln y$. Then $f' = -R$ and $f'' = -R' > 0$. Since f' is strictly increasing, integration on $[0, t]$ for $t > 0$ gives

$$f(t) > f(0) + tf'(0) = \ln 10 - \frac{14}{5}t.$$

Exponentiation preserves this inequality, so $y(t) > 10e^{-(14/5)t}$ for every $t > 0$. The extrapolation matches the initial value and slope but decays too quickly afterward. The changing mode weights, not an error in the initial data, cause the discrepancy.

Question 9

Let D denote differentiation with respect to t . Consider

$$y'' - 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = 4.$$

A student reasons that $(D - 3)(D - 1)y = 0$ means every solution must satisfy either $(D - 1)y = 0$ or $(D - 3)y = 0$.

Tasks

1. Verify the operator factorization by expanding it on a twice-differentiable function.
2. Set $z = y' - y$. Derive and solve its first-order initial-value problem.
3. Solve $y' - y = z$ using an integrating factor and the original value $y(0) = 2$. Verify the resulting solution and both initial data.
4. Test the student's either/or claim on your solution. Explain why a product of differential operators must not be interpreted as a product of two numbers acting separately on y .

Question 9 – Solution

Strategy. Factorization creates a sequence of two first-order equations, not a rule requiring one of two first-order residuals to vanish.

Step 1: Expand the operator composition. Applying $(D - 3)$ to $(D - 1)y = y' - y$ gives

$$(y' - y)' - 3(y' - y) = y'' - 4y' + 3y.$$

Thus the factorization is valid and its characteristic roots are 1 and 3.

Step 2: Solve for the intermediate quantity. With $z = y' - y$, the equation is $z' - 3z = 0$. The data imply $z(0) = 4 - 2 = 2$. Separating or using an integrating factor yields

$$\boxed{z = 2e^{3t}}.$$

The intermediate residual is nonzero, yet it is annihilated by $D - 3$.

Step 3: Recover the original solution. We must solve $y' - y = 2e^{3t}$. Multiplication by e^{-t} gives

$$(e^{-t}y)' = 2e^{2t}.$$

Integrating from 0 to t and using $y(0) = 2$ gives $e^{-t}y - 2 = e^{2t} - 1$. Hence

$$\boxed{y = e^t + e^{3t}}.$$

Indeed $y(0) = 2$, $y'(0) = 1 + 3 = 4$, and

$$y'' - 4y' + 3y = (1 - 4 + 3)e^t + (9 - 12 + 3)e^{3t} = 0.$$

The construction recovers both modes by first-order methods and gives the unique solution with the prescribed data.

Step 4: Refute the incorrect either/or inference. For this solution,

$$(D - 1)y = 2e^{3t} \neq 0, \quad (D - 3)y = -2e^t \neq 0.$$

Nevertheless, $(D - 3)((D - 1)y) = (D - 3)(2e^{3t}) = 0$. An operator composition first produces a new function and then applies another operator to it. A nonzero function can be sent to zero by differentiation combined with multiplication. The scalar zero-product rule therefore does not imply the proposed either/or claim. Sums of the two modes must be retained, not just the two separate pure-mode families.

Question 10

For $0 < \varepsilon \leq 1$, consider

$$y'' - 2y' + (1 - \varepsilon^2)y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

All parameter values in the problem have real distinct roots. You may use, for $z \geq 0$, the Taylor bound

$$0 \leq \frac{e^z - e^{-z}}{2} - z \leq \frac{z^3 e^z}{6}.$$

Tasks

1. Find the roots and the exact solution y_ε , including both mode coefficients.
2. For fixed $t \geq 0$, find the limit of $y_\varepsilon(t)$ as $\varepsilon \downarrow 0$. Explain how diverging mode coefficients can still produce a finite limit.
3. Prove an explicit bound on $|y_\varepsilon(t) - te^t|$ for $0 \leq t \leq T$, where $T > 0$ is fixed. Deduce uniform convergence on that interval.
4. Explain the numerical danger in subtracting the two close exponentials when εt is small. Derive an equivalent expression using $H(z) = (e^z - e^{-z})/(2z)$, with $H(0) = 1$, and a small- z expansion that avoids that subtraction. Do not solve the limiting differential equation as a separate root case.

Question 10 – Solution

Strategy. Keep the roots distinct while studying cancellation in the data-to-coefficient representation and a controlled limit of its solutions.

Step 1: Solve the distinct-root IVP. The characteristic polynomial is $(r - 1)^2 - \varepsilon^2$, with roots $1 + \varepsilon$, $1 - \varepsilon$. Writing their coefficients as A, B gives $A + B = 0$, $(1 + \varepsilon)A + (1 - \varepsilon)B = 1$. Thus

$$\boxed{A = 1/(2\varepsilon), \quad B = -1/(2\varepsilon)}, \quad \boxed{y_\varepsilon = \frac{e^{(1+\varepsilon)t} - e^{(1-\varepsilon)t}}{2\varepsilon}}.$$

Both modes satisfy the equation; the value and derivative at zero are 0 and 1.

Step 2: Take a function limit with cancellation intact. Factoring out e^t and using the derivative of the exponential at zero gives

$$y_\varepsilon = e^t \frac{e^{\varepsilon t} - e^{-\varepsilon t}}{2\varepsilon} \longrightarrow \boxed{te^t}.$$

At $t = 0$ the identity and limit are both zero. The coefficients grow without bound, but the two modes approach each other and their leading contributions cancel. Large coefficients in this basis do not alone imply large solution values on a fixed bounded interval.

Step 3: Control the limit on the whole finite interval. Apply the supplied bound with $z = \varepsilon t$ and multiply by e^t/ε :

$$0 \leq y_\varepsilon - te^t \leq \frac{\varepsilon^2 t^3 e^{(1+\varepsilon)t}}{6} \leq \boxed{\frac{\varepsilon^2 T^3 e^{2T}}{6}} \quad (0 \leq t \leq T).$$

The last expression tends to zero independently of t , proving uniform convergence on each fixed $[0, T]$. It is not a uniform statement over all $t \geq 0$.

Step 4: Separate algebraic cancellation from its numerical evaluation. The direct numerator subtracts two nearly equal numbers, potentially losing relative accuracy in rounded arithmetic. An equivalent expression is

$$\boxed{y_\varepsilon = te^t H(\varepsilon t)}, \quad H(z) = 1 + \frac{z^2}{6} + \frac{z^4}{120} + O(z^6) \quad (z \rightarrow 0).$$

This expansion follows by subtracting the Taylor series of e^z and e^{-z} before numerical evaluation. Evaluating H by this series for small z avoids the close-number subtraction; merely renaming the original quotient would not. The limit concerns this distinct-root family and does not require a general solution method for a repeated root.