

Question 1

A 2 kg mass stretches an ideal vertical spring by 0.196 m at static equilibrium. Use $g = 9.8 \text{ m/s}^2$ and neglect damping. Let x be downward displacement from equilibrium. At $t = 0$, the mass is 0.05 m below equilibrium and moving upward at 0.20 m/s.

Tasks

1. Determine the spring constant and derive the equation for x from Newton's law, starting with extension measured from the spring's natural length.
2. Solve the initial-value problem and find its oscillation amplitude and angular frequency.
3. Find the first positive time the mass crosses equilibrium and its signed velocity then.
4. Use mechanical energy to verify the crossing speed. Find the maximum spring extension from natural length and explain why gravity is absent from the final equation for x .

Question 1 – Solution

Strategy. Distinguish extension from natural length from displacement about equilibrium, and keep the upward initial velocity negative.

Step 1: Establish the coordinate and model. Write $z = \ell + x$ for downward extension, where $\ell = 0.196$ m. The static balance is $k\ell = mg$, so $k = 19.6/0.196 = 100$ N/m. Newton's law gives

$$2x'' = mg - k(\ell + x) = -100x, \quad x(0) = 0.05, \quad x'(0) = -0.20.$$

Thus $\omega_0 = \sqrt{k/m} = 5\sqrt{2}\text{ s}^{-1}$.

Step 2: Solve and find the amplitude. Fitting both data gives

$$x(t) = \frac{1}{20} \cos(5\sqrt{2}t) - \frac{1}{25\sqrt{2}} \sin(5\sqrt{2}t).$$

The amplitude satisfies $R^2 = x(0)^2 + [x'(0)/\omega_0]^2 = 1/400 + 1/1250 = 33/10000$. Hence $R = \sqrt{33}/100$ m. Differentiation verifies $x'' + 50x = 0$ and the two data.

Step 3: Locate the first crossing. The first zero lies in $0 < \omega_0 t < \pi/2$, where the motion is upward. Solving $x = 0$ gives

$$t_* = \frac{1}{5\sqrt{2}} \arctan \frac{5\sqrt{2}}{4}.$$

At that crossing, $x' = -\omega_0 R = -\sqrt{66}/20$ m/s. The negative sign identifies upward travel; using an arbitrary branch of arctangent could miss the first crossing.

Step 4: Check energy and physical extension. Relative to equilibrium, the conserved energy is $E = mx'^2/2 + kx^2/2$. Initially,

$$E = \frac{1}{2}(2)(0.20)^2 + \frac{1}{2}(100)(0.05)^2 = 0.165 \text{ J}.$$

At $x = 0$, $E = x'^2$, giving $|x'| = \sqrt{0.165} = \sqrt{66}/20$ m/s. The maximum natural-length extension is $\ell + R = 0.196 + \sqrt{33}/100$ m, about 0.25345 m. Gravity has not been neglected: its constant force cancels the spring's equilibrium force after shifting the coordinate.

Question 2

A mass of 1.5 kg vibrates freely about equilibrium with viscous damping:

$$mx'' + cx' + kx = 0, \quad c > 0, \quad k > 0.$$

It is released from rest at $x(0) = 0.08$ m. Measured successive positive displacement maxima are separated by π s, and each is $e^{-\pi/2}$ times the preceding one. Treat these measurements as exact.

Tasks

1. Infer the decay rate and damped angular frequency, explaining why same-sign peaks must be used. Determine c and k with units.
2. Find the response and verify that its positive maxima really have the measured spacing and ratio.
3. Determine the ratio of mechanical energies at successive positive maxima. Why is it not the displacement ratio?
4. If the mass had not been supplied, identify exactly which combinations of m, c, k these measurements determine, and exhibit the remaining ambiguity.

Question 2 – Solution

Strategy. Peak spacing determines the oscillation frequency; peak ratios determine exponential decay. A known mass converts these rates into physical coefficients.

Step 1: Infer the coefficients. An underdamped response has form $e^{-\alpha t}(A \cos \omega_d t + B \sin \omega_d t)$, with $\alpha = c/(2m)$ and $\omega_d^2 = k/m - \alpha^2$. Same-sign peaks are one damped period apart, so

$$\omega_d = \frac{2\pi}{\pi} = 2 \text{ s}^{-1}, \quad e^{-\alpha\pi} = e^{-\pi/2} \implies \alpha = \frac{1}{2} \text{ s}^{-1}.$$

Thus $\boxed{c = 3/2 \text{ N s/m}}$ and $\boxed{k = 51/8 \text{ N/m}}$. Adjacent positive and negative extrema would be only half a period apart.

Step 2: Reconstruct and verify the peaks. The initial value gives $A = 0.08$, and $x'(0) = 0$ gives $B = 0.02$. Hence

$$\boxed{x = 0.08e^{-t/2}[\cos 2t + \frac{1}{4} \sin 2t]}, \quad x' = -0.17e^{-t/2} \sin 2t.$$

The characteristic roots are $-1/2 \pm 2i$, verifying the equation. Extrema occur at $t = n\pi/2$; positive maxima occur at $t = n\pi$, where $x = 0.08e^{-n\pi/2}$. The derivative changes from positive to negative at each interior positive maximum, and the release at zero starts a decrease.

Step 3: Compare energies. At an extremum, velocity is zero, so $E = kx^2/2$. Successive positive-peak energies therefore have ratio

$$\boxed{e^{-\pi}},$$

not $e^{-\pi/2}$. Energy depends quadratically on displacement at a turning point.

Step 4: State the identification limit. Without a known mass, only $c/m = 2\alpha = 1 \text{ s}^{-1}$ and $k/m = \alpha^2 + \omega_d^2 = 17/4 \text{ s}^{-2}$ are determined. Multiplying m, c, k by the same positive constant leaves the normalized equation, the complete free response, and all peak measurements unchanged. The three coefficients cannot then be identified separately.

Question 3

A mass-spring-damper system has $m = 1$ kg, $c = 2$ Ns/m and $k = 5$ N/m. It starts at equilibrium with velocity 1 m/s in the positive direction, with no external forcing. Define

$$E(t) = \frac{1}{2}mx'(t)^2 + \frac{1}{2}kx(t)^2.$$

Tasks

1. Find the displacement and velocity, including the damping regime.
2. Derive the energy balance directly from the equation. Does decreasing energy imply that $|x|$ must decrease from the start?
3. Find the first positive turning time, its displacement, and the energy dissipated up to that time.
4. Explain why $E' = 0$ at a turning point does not mean damping has ceased permanently. Find the total energy dissipated as $t \rightarrow \infty$.

Question 3 – Solution

Strategy. Track energy in both position and velocity. Displacement can grow while total mechanical energy falls.

Step 1: Solve the free motion. The equation is $x'' + 2x' + 5x = 0$, with roots $-1 \pm 2i$, so the motion is underdamped. The data give

$$x = \frac{1}{2}e^{-t} \sin 2t, \quad x' = e^{-t}[\cos 2t - \frac{1}{2} \sin 2t].$$

These expressions have $x(0) = 0$, $x'(0) = 1$ and residual zero.

Step 2: Derive the dissipation law. Multiplying the equation by x' gives

$$E' = x'(mx'' + kx) = -cx'^2 = \boxed{-2x'^2 \leq 0}.$$

Nevertheless, $x'(0) = 1$, so x and $|x|$ initially increase from zero. Kinetic energy is partly converted into spring energy while the damper removes energy. Monotonic total energy is not monotonic displacement.

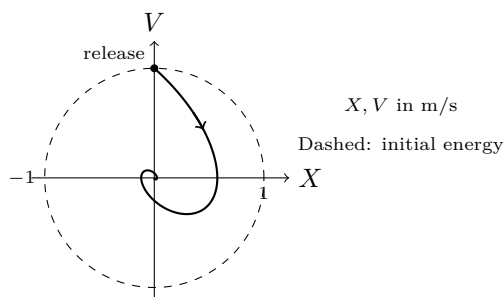
Step 3: Find the first turning point. The first root of $x' = 0$ satisfies $\tan 2t = 2$, in $0 < 2t < \pi/2$. Set $a = \frac{1}{2} \arctan 2$. Then

$$\boxed{t_* = a, \quad x(t_*) = e^{-a}/\sqrt{5} \text{ m.}}$$

The velocity changes from positive to negative, so this is a positive maximum. Initially $E(0) = 1/2 \text{ J}$. At the turning point,

$$E(a) = \frac{5}{2}x(a)^2 = \frac{1}{2}e^{-2a} \text{ J}, \quad \boxed{E(0) - E(a) = \frac{1}{2}(1 - e^{-2a}) \text{ J.}}$$

Step 4: Interpret zero instantaneous loss. At a turning point, $x' = 0$, so damper power is instantaneously zero. But $x \neq 0$ there and $x'' = -5x \neq 0$, so motion resumes immediately. The displayed response and velocity both tend to zero; hence $E \rightarrow 0$ and total dissipated energy is $1/2 \text{ J}$. In the figure, $X = \sqrt{k/m}x$ and $V = x'$ both have velocity units, and $E = m(X^2 + V^2)/2$.



Question 4

A critically damped system has $m = 1$ kg, $c = 6$ N s/m and $k = 9$ N/m. Its initial data are $x(0) = a > 0$ and $x'(0) = v$, with a in meters and v in meters per second. There is no forcing.

Tasks

1. Derive and verify the response for arbitrary a, v .
2. Classify exactly which initial velocities cause a finite equilibrium crossing. Find the crossing time when one exists.
3. Classify exactly which initial velocities make the displacement stay positive and decrease monotonically toward zero.
4. For $a = 0.10$ m and $v = -0.60$ m/s, find the crossing and the subsequent minimum. Use the result to assess the claim that critical damping always prevents overshoot.

Question 4 – Solution

Strategy. Critical damping fixes the repeated root, but the sign of the linear factor still depends on the initial velocity.

Step 1: Fit the repeated-root solution. The characteristic polynomial is $(r + 3)^2$. Thus

$$x(t) = e^{-3t}[a + (v + 3a)t], \quad x'(t) = e^{-3t}[v - 3(v + 3a)t].$$

Here the numerical rate 3 has units s^{-1} . The two data follow immediately, and substitution gives $x'' + 6x' + 9x = 0$.

Step 2: Classify crossings. Since $e^{-3t} > 0$, a positive-time zero occurs exactly when $v + 3a < 0$. In that case it is unique and

$$t_c = \frac{a}{-v - 3a}.$$

The linear factor changes sign there, so this is a crossing, not a touch. If $v \geq -3a$, the factor remains positive for all finite $t \geq 0$, including the equality case $x = ae^{-3t}$.

Step 3: Classify monotone positive return. For $v \geq -3a$, the displacement is positive. Its derivative is nonpositive for all $t \geq 0$ exactly when $v \leq 0$, because $v - 3(v + 3a)t$ then begins nonpositive and decreases. Thus the required range is

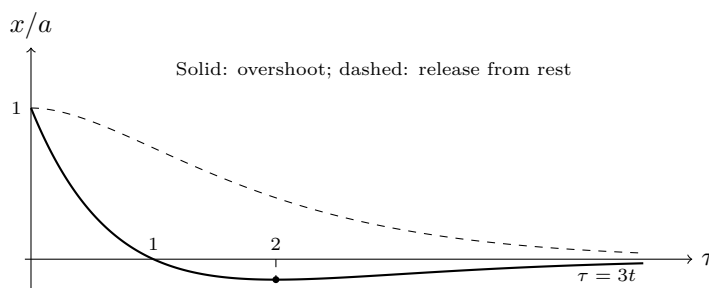
$$-3a \leq v \leq 0.$$

For $v > 0$, the mass first moves farther from equilibrium. For $v < -3a$, it crosses equilibrium and cannot be a positive monotone return.

Step 4: Exhibit overshoot. For the stated data, $x = 0.10e^{-3t}(1 - 3t)$ and $x' = e^{-3t}(-0.60 + 0.90t)$. Therefore

$$t_c = 1/3 \text{ s}, \quad t_{\min} = 2/3 \text{ s}, \quad x_{\min} = -0.10e^{-2} \text{ m}.$$

The derivative changes from negative to positive at the minimum. Critical damping eliminates sustained oscillations, but sufficiently negative initial velocity still causes one overshoot. The comparison uses $\tau = 3t$ and normalized displacement x/a .



Question 5

A 1 kg mass on a spring with $k = 4 \text{ N/m}$ is released from $x(0) = a = 0.01 \text{ m}$, with $x'(0) = 0$. Compare viscous damping coefficients $c = 4$ and $c = 5 \text{ Ns/m}$; write their responses as x_4, x_5 .

Tasks

1. Determine the damping regimes and find both exact responses.
2. Prove that each response stays positive and strictly decreases for $t > 0$.
3. Prove $x_5(t) > x_4(t)$ for every $t > 0$. You may form the equation for $w = x_5 - x_4$ and derive its zero-data integral response.
4. Explain the implication for the first time the displacement reaches any fixed level ε with $0 < \varepsilon < a$. For general $c > 4$, examine the slow characteristic root as $c \rightarrow \infty$.

Question 5 – Solution

Strategy. Compare complete release responses, rather than assuming a larger damping force always gives a faster return.

Step 1: Solve both release problems. For $c = 4$, the repeated root is -2 , giving critical damping. For $c = 5$, roots $-1, -4$ give overdamping. Fitting the data yields

$$x_4 = a(1 + 2t)e^{-2t}, \quad x_5 = \frac{a}{3}(4e^{-t} - e^{-4t}).$$

Their characteristic forms verify the equations; both have value a and slope zero at the release.

Step 2: Check positivity and decrease. The first expression is positive, and the second is positive because $4e^{-t} > e^{-4t}$. Their derivatives are

$$x'_4 = -4ate^{-2t} < 0, \quad x'_5 = \frac{4a}{3}(e^{-4t} - e^{-t}) < 0 \quad (t > 0).$$

Both tend to zero.

Step 3: Prove the comparison for every time. Applying $D^2 + 5D + 4$ to $w = x_5 - x_4$ gives

$$w'' + 5w' + 4w = -x'_4, \quad w(0) = w'(0) = 0.$$

Variation of parameters with e^{-t}, e^{-4t} yields

$$w(t) = \int_0^t \frac{e^{-(t-s)} - e^{-4(t-s)}}{3} [-x'_4(s)] ds.$$

For $0 < s < t$, both factors are strictly positive. Hence $\boxed{x_5(t) > x_4(t)}$ for every $t > 0$. This integral proof avoids relying on a finite plot or only on large-time rates.

Step 4: Interpret settling and heavy damping. Each response reaches any $\varepsilon \in (0, a)$ exactly once, by continuity and strict decrease. At the time $x_4 = \varepsilon$, one still has $x_5 > \varepsilon$, so the more heavily damped response reaches the level later. For general $c > 4$ in this SI model,

$$r_{\text{slow}} = \frac{-c + \sqrt{c^2 - 4mk}}{2m} = \frac{-2k}{c + \sqrt{c^2 - 4mk}} \rightarrow 0^-.$$

The release excites this mode with coefficient $-ar_{\text{fast}}/(r_{\text{slow}} - r_{\text{fast}}) > 0$. Excessive damping therefore introduces a slow return, with $r_{\text{slow}} \sim -k/c$; damping strength alone is not a measure of settling speed.

Question 6

An undamped 1 kg mass on a 25 N/m spring is initially at rest at equilibrium. For $t \geq 0$, apply the force $2 \cos 4t$ N.

Tasks

1. Derive and solve the forced initial-value problem.
2. Express the response as a product that displays beats. Identify the carrier angular frequency and the period of the nonnegative amplitude envelope.
3. At $t = 2\pi$ s, find both displacement and velocity. Does this mean the mass stays at rest afterward? Justify your answer using the equation.
4. Replace the forcing by $2 \cos 5t$ N and find the new zero-data response. Explain the difference between beats and exact resonance.

Question 6 – Solution

Strategy. The homogeneous correction needed to fit rest data creates the nearby-frequency interference; exact resonance requires a different particular form.

Step 1: Solve the nonresonant problem. Newton's law gives $x'' + 25x = 2 \cos 4t$, $x(0) = x'(0) = 0$. Since $25 - 16 = 9$, a particular response is $(2/9) \cos 4t$. Fitting zero data gives

$$x = \frac{2}{9}(\cos 4t - \cos 5t).$$

Its second derivative plus $25x$ equals $2 \cos 4t$, and both initial data vanish.

Step 2: Separate the two time scales. The cosine-difference identity gives

$$x = \frac{4}{9} \sin(9t/2) \sin(t/2).$$

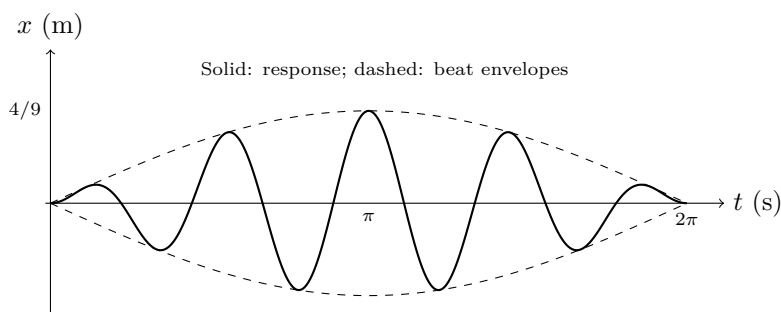
The carrier angular frequency is $9/2 \text{ s}^{-1}$. The nonnegative envelope is $A(t) = (4/9)|\sin(t/2)| \text{ m}$, with period $2\pi \text{ s}$. Its zeros occur at $t = 2n\pi$, and its largest value is $4/9 \text{ m}$, attained by x at $t = \pi$. The envelope period uses the absolute value; the signed slow sine has period 4π .

Step 3: Distinguish a rest state from remaining at rest. Here $x' = \frac{2}{9}(-4 \sin 4t + 5 \sin 5t)$, so $x(2\pi) = x'(2\pi) = 0$. Yet the applied force then equals 2 N ; the equation gives $x''(2\pi) = 2 \text{ m/s}^2$. The mass immediately accelerates again. Reaching the equilibrium rest state while a nonzero force persists does not keep it there.

Step 4: Solve at exact resonance. For $x'' + 25x = 2 \cos 5t$, use $x_p = At \sin 5t$. Its residual is $10A \cos 5t$, so $A = 1/5$ and the data are already zero:

$$x_{\text{res}} = \frac{1}{5}t \sin 5t.$$

At $t_n = (\pi/2 + 2n\pi)/5$, it equals $t_n/5 \rightarrow \infty$, whereas the beating response is bounded by $4/9$. Beats arise from interference of distinct frequencies; resonance produces a growing factor when the forcing matches the natural frequency.



Question 7

A mass-spring-damper system has $m = 2$ kg, $c = 4$ N s/m and $k = 20$ N/m. Forcing is $F_0 \cos \omega t$, where $F_0 > 0$ is fixed and $\omega > 0$. Consider its steady periodic response. Let X be displacement amplitude and V be velocity amplitude.

Tasks

1. Derive the cosine and sine coefficients of the steady response and express X and V in terms of ω .
2. Find the frequency that maximizes displacement amplitude and its maximum value.
3. Independently find the frequency that maximizes velocity amplitude and its maximum value.
4. At the velocity-maximizing frequency, determine the displacement phase relative to the applied force. Explain why a request for the 'resonant frequency' needs a specified response quantity.

Question 7 – Solution

Strategy. Different measured quantities introduce different frequency factors; optimize their amplitudes separately.

Step 1: Match harmonic coefficients. Set $x = A \cos \omega t + B \sin \omega t$. The equation $2x'' + 4x' + 20x = F_0 \cos \omega t$ gives

$$(20 - 2\omega^2)A + 4\omega B = F_0, \quad -4\omega A + (20 - 2\omega^2)B = 0.$$

Writing $\Delta = (20 - 2\omega^2)^2 + 16\omega^2 > 0$, we obtain

$$A = \frac{F_0(20 - 2\omega^2)}{\Delta}, \quad B = \frac{4F_0\omega}{\Delta}, \quad \boxed{X = \frac{F_0}{\sqrt{\Delta}}, \quad V = \omega X.}$$

The homogeneous roots are $-1 \pm 3i$, so all initial-data transients decay and this periodic response is the long-time motion.

Step 2: Maximize displacement. Complete the square:

$$\Delta = 4[(\omega^2 - 8)^2 + 36].$$

It has its unique minimum for $\omega > 0$ at $\omega_X = 2\sqrt{2}\text{s}^{-1}$. Therefore $\boxed{X_{\max} = F_0/12}$ in meters when F_0 is expressed in newtons.

Step 3: Maximize velocity. Divide the amplitude denominator by ω :

$$V = \frac{F_0}{\sqrt{(20/\omega - 2\omega)^2 + 16}}.$$

This is largest exactly when $20/\omega - 2\omega = 0$, giving

$$\boxed{\omega_V = \sqrt{10}\text{s}^{-1}, \quad V_{\max} = F_0/4}$$

in meters per second when F_0 is in newtons. The spring and inertial contributions to this denominator cancel there.

Step 4: Interpret the phase and the two maxima. At $\omega = \sqrt{10}$, $A = 0$ and $B = F_0/(4\sqrt{10}) > 0$. Thus $x = [F_0/(4\sqrt{10})] \sin(\sqrt{10}t)$, a displacement lag of $\pi/2$ behind the cosine force. Its displacement amplitude is smaller than $F_0/12$, even though its velocity amplitude is largest. The displacement peak, velocity peak and undamped natural frequency need not all coincide; the response quantity must be named.

Question 8

A 1 kg mass is connected to a moving support by a spring with $k = 9 \text{ N/m}$ and a parallel viscous damper with $c = 2 \text{ N s/m}$. Let $b(t) = B \cos \omega t$ be support displacement and $x(t)$ the mass's absolute displacement, measured in the same positive direction about equilibrium. Here $B > 0$ and $\omega > 0$. Spring and damping forces depend on relative displacement and relative velocity.

Tasks

1. Derive the equations for absolute displacement x and relative displacement $z = x - b$.
2. Find the steady amplitude ratios X/B and Z/B , where X and Z are the amplitudes of x and z .
3. Determine exactly when the absolute amplitude is smaller than the support amplitude, including the equality frequency.
4. Find the low- and high-frequency limits of both ratios. Explain why good isolation of the mass does not imply small relative spring deformation.

Question 8 – Solution

Strategy. The damper responds to velocity relative to the support. Distinguish motion of the mass from deformation of the suspension.

Step 1: Derive the two models. Newton's law gives $x'' = -9(x - b) - 2(x' - b')$. Therefore

$$x'' + 2x' + 9x = 2b' + 9b, \quad \boxed{z'' + 2z' + 9z = -b'' = B\omega^2 \cos \omega t.}$$

The inertial term in the relative equation comes from $x'' = z'' + b''$; it is not an extra spring force.

Step 2: Compute both amplitude ratios. Let $\Delta = (9 - \omega^2)^2 + 4\omega^2$. Matching sine and cosine in the relative equation gives

$$z = A \cos \omega t + D \sin \omega t, \quad A = \frac{B\omega^2(9 - \omega^2)}{\Delta}, \quad D = \frac{2B\omega^3}{\Delta}.$$

Thus $Z/B = \omega^2/\sqrt{\Delta}$. Since $x = z + B \cos \omega t$, its cosine coefficient is $B + A$, not its amplitude $B + Z$. Simplifying $(B + A)^2 + D^2$ gives

$$\boxed{\frac{X}{B} = \frac{\sqrt{81 + 4\omega^2}}{\sqrt{\Delta}}, \quad \frac{Z}{B} = \frac{\omega^2}{\sqrt{\Delta}}.}$$

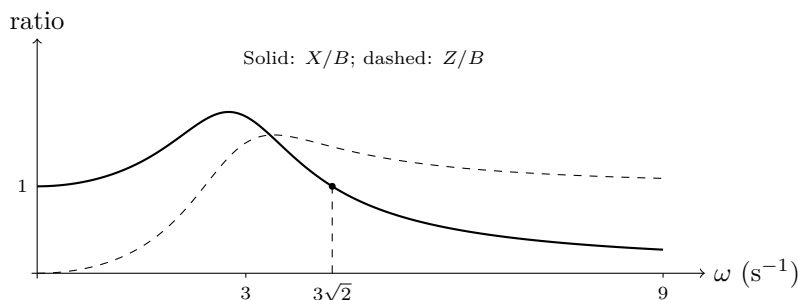
The homogeneous roots $-1 \pm 2\sqrt{2}i$ ensure the initial-data transients decay.

Step 3: Find the isolation threshold. Since $\Delta > 0$, $X/B < 1$ exactly when

$$\Delta - (81 + 4\omega^2) = \omega^2(\omega^2 - 18) > 0.$$

For $\omega > 0$, this means $\boxed{\omega > 3\sqrt{2} \text{ s}^{-1}}$. Equality holds at $3\sqrt{2}$; below that positive frequency the absolute motion is amplified.

Step 4: Interpret the limiting motions. As $\omega \rightarrow 0^+$, $X/B \rightarrow 1$ and $Z/B \rightarrow 0$: the mass follows the slowly moving support. As $\omega \rightarrow \infty$, $X/B \rightarrow 0$ but $Z/B \rightarrow 1$. In fact $A/B \rightarrow -1$ and $D/B \rightarrow 0$, so $z \approx -b$ while $x \approx 0$. An almost stationary mass can still have relative deformation comparable to the base motion.



Question 9

A 1 kg mass with $c = 2 \text{ N s/m}$ and $k = 5 \text{ N/m}$ obeys $x'' + 2x' + 5x = F(t)$ in SI units. An actuator must move it from $x(0) = 0.10 \text{ m}$, $x'(0) = 0$ to $x(1) = x'(1) = 0$, then turn off for $t > 1$. Before $t = 0$, it is held motionless at $x = 0.10 \text{ m}$ by a constant force.

Tasks

1. Find the unique cubic trajectory satisfying the four prescribed endpoint data on $[0, 1]$.
2. Compute its required force. Examine the force and acceleration when it joins the held state at zero and the rest state at one.
3. Instead require zero acceleration at both endpoints as well. Find the unique polynomial trajectory of degree at most five satisfying all six conditions.
4. Compute the new force and verify that it joins the pre-motion holding force and the zero post-motion force continuously. Show that the mass remains at rest after one second.

Question 9 – Solution

Strategy. Position and velocity matching ensure a continuous state, but continuous actuator force also requires compatible acceleration.

Step 1: Fit the cubic. Write $x = a_0 + a_1t + a_2t^2 + a_3t^3$. The initial conditions give $a_0 = 0.10$, $a_1 = 0$; the final conditions give $a_2 + a_3 = -0.10$, $2a_2 + 3a_3 = 0$. Hence

$$x_c = 0.10(1 - 3t^2 + 2t^3).$$

The nonsingular two-equation system makes this cubic unique.

Step 2: Check the actuator joins. Differentiating gives $x'_c = 0.60(t^2 - t)$ and $x''_c = 1.20t - 0.60$. Thus

$$F_c = x''_c + 2x'_c + 5x_c = t^3 - 0.30t^2 - 0.10 \quad \text{N.}$$

The holding force before zero is $5(0.10) = 0.50$ N. Yet $F_c(0) = -0.10$ N and $F_c(1) = 0.60$ N, so both joins require force jumps. Acceleration jumps from zero to -0.60 initially and from 0.60 to zero finally. The joined trajectory is C^1 and piecewise C^2 , not globally C^2 .

Step 3: Add acceleration matching. For a polynomial of degree at most five, the initial conditions fix $a_0 = 0.10$, $a_1 = a_2 = 0$. The endpoint conditions become

$$a_3 + a_4 + a_5 = -0.10, \quad 3a_3 + 4a_4 + 5a_5 = 0, \quad 6a_3 + 12a_4 + 20a_5 = 0.$$

Solving gives $a_3 = -1$, $a_4 = 1.5$, $a_5 = -0.6$, uniquely. Therefore

$$x_q = 0.10(1 - 10t^3 + 15t^4 - 6t^5).$$

Step 4: Verify force continuity and rest. Substitution into the model yields

$$F_q = 0.50 - 6t + 12t^2 - 5t^3 + 1.5t^4 - 3t^5 \quad \text{N.}$$

It has $F_q(0) = 0.50$ and $F_q(1) = 0$, matching the constant forces outside the motion interval. All six trajectory conditions hold, so the joined motion is C^2 . For $t > 1$, zero forcing and zero value and slope at one have the unique solution $x = 0$. The improvement is continuous force and acceleration; no claim of a force-minimizing trajectory is needed.

Question 10

A 1 kg mass with damping $c = 2 \text{ N s/m}$ and spring constant $k = 9 \text{ N/m}$ is driven by $F(t) = 6 \cos \omega t \text{ N}$, with $\omega > 0$. Consider only the steady periodic response. For a periodic quantity h , define its cycle average by

$$\langle h \rangle = \frac{1}{T} \int_0^T h(t) dt, \quad T = 2\pi/\omega.$$

Tasks

1. Derive the instantaneous energy balance and the relation between average input power and average damper dissipation.
2. Find the steady response coefficients and derive an explicit formula for average input power as a function of ω .
3. Find the unique positive frequency maximizing average input power and the maximum power.
4. At that frequency, find the instantaneous response, stored mechanical energy, and work supplied in one period. Explain how positive work per cycle can coexist with constant stored energy.

Question 10 – Solution

Strategy. The work done by the actuator splits into stored energy and damping loss. Periodicity removes the net storage change over a cycle.

Step 1: Derive the power balance. For $E = x'^2/2 + 9x^2/2$, multiplying $x'' + 2x' + 9x = F$ by x' gives

$$E' = Fx' - 2x'^2.$$

The steady energy is periodic, so integrating over one period gives $\langle Fx' \rangle = 2\langle x'^2 \rangle$. This is an average statement for general ω .

Step 2: Compute the mean input. Write $x = A \cos \omega t + B \sin \omega t$. Coefficient matching yields

$$\Delta = (9 - \omega^2)^2 + 4\omega^2, \quad A = \frac{6(9 - \omega^2)}{\Delta}, \quad B = \frac{12\omega}{\Delta}.$$

Since $x' = -\omega A \sin \omega t + \omega B \cos \omega t$, averaging the products gives

$$\bar{P} = \langle 6 \cos \omega t x' \rangle = 3\omega B = \frac{36\omega^2}{\Delta} \quad \text{W.}$$

Also $2\langle x'^2 \rangle = \omega^2(A^2 + B^2) = 36\omega^2/\Delta$, independently checking the energy balance.

Step 3: Maximize average power. For $\omega > 0$,

$$\bar{P} = \frac{36}{(9/\omega - \omega)^2 + 4} \leq 9 \text{ W.}$$

Equality holds uniquely when $\omega^2 = 9$. Thus $\omega = 3 \text{ s}^{-1}$, $\bar{P}_{\max} = 9 \text{ W}$.

Step 4: Account for the work at the maximum. At this frequency $A = 0$, $B = 1$, so $x = \sin 3t \text{ m}$ and $x' = 3 \cos 3t \text{ m/s}$. Therefore

$$E = \frac{9}{2}(\cos^2 3t + \sin^2 3t) = \boxed{4.5 \text{ J}}.$$

The instantaneous input and damping powers are both $18 \cos^2 3t \text{ W}$: here $E' = 0$ at every instant, not just on average. Over $T = 2\pi/3 \text{ s}$, the supplied work is $9T = \boxed{6\pi \text{ J}}$, exactly the energy dissipated. Constant stored energy does not mean zero energy throughput.

