

Question 1

On $t > 0$, consider

$$t^2 y'' - 2ty' + 2y = t^3, \quad y(1) = y'(1) = 0.$$

Two homogeneous solutions are $y_1 = t$ and $y_2 = t^2$.

Tasks

1. Normalize the equation and compute the Wronskian. Derive the two equations for the parameter derivatives using the auxiliary condition $u_1' y_1 + u_2' y_2 = 0$.
2. Use variation of parameters to find a particular solution and then fit the initial data.
3. Check the final solution in the original, unnormalized equation and at the initial point.
4. A student inserts t^3 , rather than the normalized forcing, into the parameter formulas. Find the resulting particular expression and its actual forcing in the original equation. Explain the error.

Question 1 – Solution

Strategy. Divide by the leading coefficient before solving for the parameter derivatives; check the answer in the original equation.

Step 1: Normalize and derive. The normalized forcing is $g(t) = t$, and $W = t(2t) - t^2 = t^2 \neq 0$. With $y = u_1t + u_2t^2$, impose $tu'_1 + t^2u'_2 = 0$. Differentiating after this cancellation gives

$$tu'_1 + t^2u'_2 = 0, \quad u'_1 + 2tu'_2 = t.$$

Thus $u'_1 = -t$ and $u'_2 = 1$.

Step 2: Integrate and fit. Choose $u_1 = -t^2/2$, $u_2 = t$; constants of integration contribute only homogeneous terms. Hence $y_p = t^3/2$ and

$$y = Ct + Dt^2 + t^3/2.$$

The data give $C + D = -1/2$ and $C + 2D = -3/2$, so $C = 1/2$, $D = -1$. Therefore

$$y = \frac{1}{2}t(t-1)^2.$$

Step 3: Verify the original equation. For the operator $L[y] = t^2y'' - 2ty' + 2y$, one has $L[t] = L[t^2] = 0$ and $L[t^3] = 2t^3$. Thus $L[y] = t^3$. Also $y(1) = 0$ and $y' = (3t^2 - 4t + 1)/2$, so $y'(1) = 0$. The normalized coefficients are continuous on $(0, \infty)$, giving uniqueness there.

Step 4: Diagnose the missing division. Using $g = t^3$ instead gives $u'_1 = -t^3$, $u'_2 = t^2$ and

$$\tilde{y}_p = -t^5/4 + t^5/3 = t^5/12.$$

Since $L[t^5] = (20 - 10 + 2)t^5 = 12t^5$, this expression satisfies $L[\tilde{y}_p] = t^5$, not t^3 . Adding homogeneous terms cannot repair its forcing. The parameter formulas apply to the normalized equation; omitting the division multiplies the intended original forcing by t^2 .

Question 2

Solve on $I = (-\pi/2, \pi/2)$:

$$y'' + y = \sec t, \quad y(0) = y'(0) = 0.$$

Use the homogeneous pair $\cos t, \sin t$.

Tasks

1. Derive the parameter derivatives and evaluate the integrals with lower endpoint zero.
2. Simplify the initial-value solution and verify its equation and both data.
3. Determine the limits of y and y' as $t \rightarrow (\pi/2)^-$. Can the solution extend as a C^1 function through that endpoint?
4. Explain why a finite limit of y does not enlarge the interval on which this initial-value problem has a classical solution.

Question 2 – Solution

Strategy. The forcing is singular at the interval endpoints. Keep its domain visible even when the solution itself has a finite limit.

Step 1: Integrate the parameter system. Here $W = 1$. The equations $u_1' \cos t + u_2' \sin t = 0$ and $-u_1' \sin t + u_2' \cos t = \sec t$ give

$$u_1' = -\tan t, \quad u_2' = 1.$$

Since $\cos t > 0$ on I , integration from zero yields $u_1 = \ln(\cos t)$, $u_2 = t$.

Step 2: Verify the response. The zero lower endpoints fit both data, and

$$y = \cos t \ln(\cos t) + t \sin t.$$

Differentiation simplifies to

$$y' = -\sin t \ln(\cos t) + t \cos t, \quad y'' = -\cos t \ln(\cos t) + \frac{\sin^2 t}{\cos t} + \cos t - t \sin t.$$

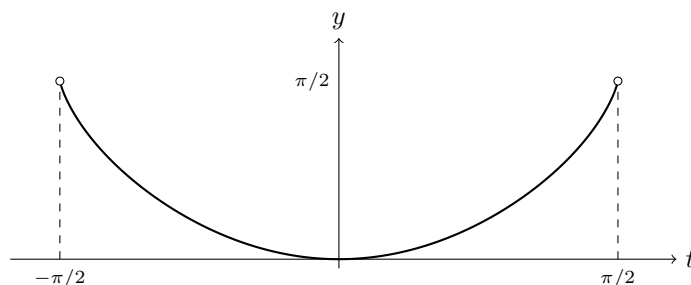
Therefore $y'' + y = (\sin^2 t + \cos^2 t)/\cos t = \sec t$, and both data vanish.

Step 3: Examine the endpoint. Using $x \ln x \rightarrow 0$ as $x \rightarrow 0^+$, we obtain

$$\lim_{t \rightarrow (\pi/2)^-} y(t) = \pi/2, \quad \lim_{t \rightarrow (\pi/2)^-} y'(t) = +\infty.$$

Thus even a C^1 extension is impossible. The graph approaches a finite height with an unbounded positive slope; the endpoint is excluded.

Step 4: Keep the equation's domain. The right side $\sec t$ is undefined at $\pi/2$ and $-\pi/2$. A continuous assignment to y at an endpoint would not define the differential equation there. The maximal open interval containing zero on which this IVP is classical is exactly I .



Question 3

On $\mathbb{R}$, consider

$$y'' - 2y' + y = \frac{e^t}{1+t^2}, \quad y(0) = y'(0) = 0.$$

Use $y_1 = e^t$ and $y_2 = te^t$.

Tasks

1. Compute the Wronskian and use variation of parameters to solve the initial-value problem.
2. Independently verify the result by writing $y = e^t F(t)$ and simplifying the differential equation for F .
3. Prove that the selected response is strictly positive for every $t > 0$.
4. Find $\lim_{t \rightarrow \infty} y(t)/(te^t)$ and interpret the effect of the factor $1/(1+t^2)$ in the forcing.

Question 3 – Solution

Strategy. The repeated homogeneous root is handled by the Wronskian; an exponential substitution provides an independent residual check.

Step 1: Compute and integrate. The Wronskian is $e^t(1+t)e^t - te^{2t} = e^{2t}$. Hence

$$u_1' = -\frac{t}{1+t^2}, \quad u_2' = \frac{1}{1+t^2}.$$

Integrating from zero gives $u_1 = -\frac{1}{2}\ln(1+t^2)$ and $u_2 = \arctan t$. Therefore

$$y = e^t[t \arctan t - \frac{1}{2}\ln(1+t^2)].$$

Zero integration endpoints ensure $y(0) = y'(0) = 0$.

Step 2: Check by a different calculation. Writing $y = e^t F$ gives $y'' - 2y' + y = e^t F''$. For the displayed bracket,

$$F' = \arctan t, \quad F'' = \frac{1}{1+t^2}, \quad F(0) = F'(0) = 0.$$

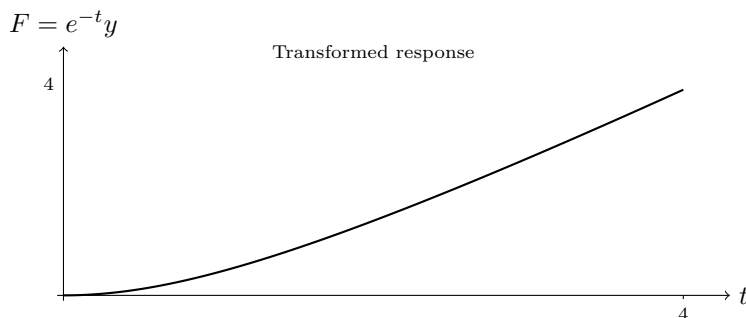
These identities verify the equation and both data independently. All coefficients and the forcing are continuous on $\mathbb{R}$, so this is the unique global IVP solution.

Step 3: Prove the sign. For $t > 0$, $F(t) = \int_0^t \arctan s \, ds > 0$. Equivalently, $F(t) = \int_0^t (t-s)/(1+s^2) \, ds$ has a positive integrand in the interior. Since $e^t > 0$, the response is strictly positive.

Step 4: Compare the growth rate. For $t > 0$,

$$\frac{y}{te^t} = \arctan t - \frac{\ln(1+t^2)}{2t} \rightarrow \boxed{\pi/2}.$$

Although the forcing is smaller than e^t by a factor tending to zero, the zero-data response is asymptotic to $(\pi/2)te^t$. The graph shows the transformed response $F = e^{-t}y$, whose slope tends to $\pi/2$; it is not a plot of y itself.



Question 4

Let g be continuous on $\mathbb{R}$. For $y'' + y = g(t)$, compare the homogeneous bases

$$(y_1, y_2) = (\cos t, \sin t), \quad (z_1, z_2) = (\cos t + \sin t, \cos t - \sin t).$$

In both parameter constructions, set the two parameter values to zero at $t = 0$.

Tasks

1. Derive the parameter derivatives for both bases, including the sign of each Wronskian.
2. Prove directly that both constructions give the same integral formula for the zero-data response.
3. If the lower endpoint is changed from zero to a , express the difference between the two particular solutions as a homogeneous combination.
4. Apply the formula to $g(t) = e^{t^2}$ with $y(0) = 2$, $y'(0) = -1$. Verify the solution without assuming that its integrals have elementary antiderivatives.

Question 4 – Solution

Strategy. A basis changes the parameter functions, but the integral response with fixed initial data must remain the same.

Step 1: Compute both parameter systems. For the first basis $W = 1$, so $u'_1 = -g \sin t$, $u'_2 = g \cos t$. The constant change-of-basis determinant is -2 , hence $W_z = -2$. Thus

$$v'_1 = \frac{1}{2}g(\cos t - \sin t), \quad v'_2 = -\frac{1}{2}g(\cos t + \sin t).$$

The negative Wronskian is essential to these signs.

Step 2: Compare the reconstructed functions. For the first basis, the integrand multiplying $g(s)$ is $\sin t \cos s - \cos t \sin s = \sin(t - s)$. For the second, it is

$$\frac{1}{2}[(\cos t + \sin t)(\cos s - \sin s) - (\cos t - \sin t)(\cos s + \sin s)],$$

which expands to the same expression. Therefore both give

$$Y_0(t) = \int_0^t \sin(t-s)g(s) ds.$$

This also holds for negative t using oriented integrals.

Step 3: Change the lower endpoint. Set $Y_a(t) = \int_a^t \sin(t-s)g(s) ds$. Then

$$Y_a - Y_0 = \cos t \int_0^a \sin s g(s) ds - \sin t \int_0^a \cos s g(s) ds.$$

Both coefficients are constants in t . Changing the lower endpoint changes a particular solution only by a homogeneous term; the data must be fitted again.

Step 4: Use a nonelementary forcing. The requested solution is

$$y = 2 \cos t - \sin t + \int_0^t \sin(t-s)e^{s^2} ds.$$

Leibniz differentiation gives $Y'_0 = \int_0^t \cos(t-s)e^{s^2} ds$ and $Y''_0 = e^{t^2} - Y_0$. Also $Y_0(0) = Y'_0(0) = 0$. Thus the equation and data hold. A definite-integral formula is a complete exact solution even without an elementary antiderivative.

Question 5

On $t > 0$, solve

$$ty'' - (t+1)y' + y = t^2, \quad y(1) = y'(1) = 0.$$

The functions $t+1$ and e^t solve the homogeneous equation. Also consider the original undivided equation at $t=0$.

Tasks

1. Normalize the equation, compute the Wronskian, and obtain a particular solution by variation of parameters.
2. Fit the initial data and check the result directly in the undivided equation.
3. Does this selected function extend smoothly through zero and still satisfy the undivided equation? Find its value and slope there.
4. Prove that these same value and slope at zero do not select a unique smooth solution of the undivided equation near zero.

Question 5 – Solution

Strategy. Use variation of parameters where the leading coefficient is nonzero, then examine the original equation separately at its singular point.

Step 1: Normalize and integrate. The normalized forcing is $g = t$, and $W = (t+1)e^t - e^t = te^t$. Hence

$$u_1' = -1, \quad u_2' = (t+1)e^{-t}.$$

Choose $u_1 = -t$, $u_2 = -(t+2)e^{-t}$, giving

$$y_p = -t(t+1) - (t+2) = -t^2 - 2t - 2.$$

Thus $y = C(t+1) + De^t - t^2 - 2t - 2$ on $(0, \infty)$.

Step 2: Fit and verify. The conditions are $2C + De = 5$ and $C + De = 4$, so $C = 1$, $D = 3/e$ and

$$y = 3e^{t-1} - t^2 - t - 1.$$

Its derivatives are $y' = 3e^{t-1} - 2t - 1$, $y'' = 3e^{t-1} - 2$. Substitution cancels the exponential terms; the polynomial terms yield

$$-2t + (t+1)(2t+1) - t^2 - t - 1 = t^2.$$

Both data at one vanish. The normalized problem is regular on $(0, \infty)$, where its solution is unique.

Step 3: Continue the selected function. The formula is analytic on $\mathbb{R}$, and the same residual identity holds there. At zero,

$$y(0) = y'(0) = 3/e - 1.$$

The undivided equation reduces to $-y'(0) + y(0) = 0$, which these values satisfy. This extension does not require dividing by t at zero.

Step 4: Demonstrate nonuniqueness at zero. The nonzero homogeneous function $h = e^t - (t+1)$ is analytic, with $h(0) = h'(0) = 0$. For every real k , $y + kh$ solves the undivided equation and has the same two data at zero. These are distinct functions since $h''(0) = 1$. The Wronskian vanishes at zero precisely where the leading coefficient vanishes; the regular normalized uniqueness theorem does not apply there.

Question 6

Let $T > 0$ and let g be continuous on $[0, T]$. Consider

$$y'' - y = g(t), \quad y(0) = y'(0) = 0.$$

Use the homogeneous basis e^t, e^{-t} .

Tasks

1. Derive a single definite-integral formula for the response by variation of parameters, and verify it by differentiation.
2. If $|g(t)| \leq M$ with $M > 0$, find a sharp bound for $|y(t)|$ on $[0, T]$ and exhibit forcing that attains it.
3. If $g \geq 0$, prove $y \geq 0$ and give a precise condition guaranteeing $y(t) > 0$ at a specified time $t > 0$.
4. For $g(t) = (1 + \cos 2t)/2$, find an elementary response and compare it with the bound for $M = 1$.

Question 6 – Solution

Strategy. The sign of the integral kernel gives comparison and error bounds without evaluating each forcing integral.

Step 1: Derive the response kernel. Here $W = -2$, so $u'_1 = e^{-t}g/2$ and $u'_2 = -e^tg/2$. Integrating from zero and combining gives

$$y(t) = \int_0^t \sinh(t-s)g(s) ds.$$

Differentiation gives $y' = \int_0^t \cosh(t-s)g(s) ds$ and $y'' = g(t) + y(t)$. Both initial data vanish.

Step 2: Prove sharpness. For $0 \leq s \leq t$, $\sinh(t-s) \geq 0$. Consequently

$$|y(t)| \leq M \int_0^t \sinh(t-s) ds = M(\cosh t - 1).$$

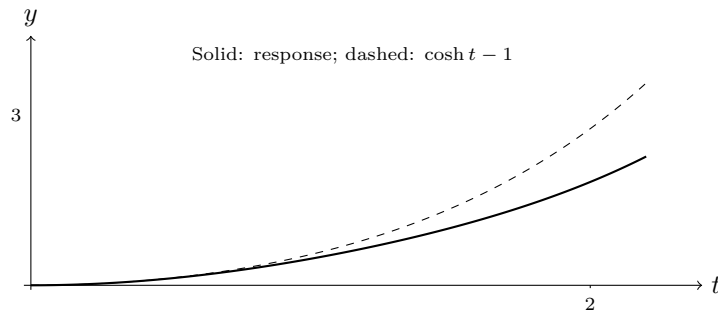
Constant forcing $g = M$ gives equality for every t , and $g = -M$ gives equality in magnitude. The bound is therefore sharp for this forcing class; it also bounds response error when g denotes a forcing error.

Step 3: Prove positivity carefully. If $g \geq 0$, the integral is nonnegative. At fixed $t > 0$, it is strictly positive exactly when g is not identically zero on $[0, t]$. Indeed, continuity then gives a positive subinterval lying before t , where the kernel is strictly positive. Forcing that is positive only at later times cannot make the response positive at this time.

Step 4: Evaluate a comparison example. For $g = (1 + \cos 2t)/2$, an elementary particular solution is $-1/2 - \cos(2t)/10$. Fitting zero data gives

$$y = \frac{3}{5} \cosh t - \frac{1}{2} - \frac{1}{10} \cos 2t.$$

Direct differentiation yields $y'' - y = (1 + \cos 2t)/2$ and zero data. Since $0 \leq g \leq 1$, the kernel proves $0 < y(t) < \cosh t - 1$ for $t > 0$: both g and $1 - g$ are nonzero on every such interval.



Question 7

On $t \geq 0$, consider the family

$$y'' - y = \frac{1}{1+t^2}, \quad y(0) = 0, \quad y'(0) = v.$$

Write $f(t) = 1/(1+t^2)$ and $I = \int_0^\infty e^{-s} f(s) ds$. An answer containing a convergent definite integral is exact.

Tasks

1. Use variation of parameters to express y_v as an integral. Prove that I converges.
2. Find the only possible value of v for which y_v is bounded on $[0, \infty)$.
3. For that value, rewrite the response using the positive kernel $K(t, s) = \sinh(\min\{t, s\})e^{-\max\{t, s\}}$. Prove boundedness and determine its sign for $t > 0$.
4. Prove that the bounded response tends to zero, and determine the long-time behavior when the initial slope differs from the selected value.

Question 7 – Solution

Strategy. Cancel the growing mode using a convergent integral, then prove the resulting tail expression is bounded.

Step 1: Construct the family. Variation of parameters with e^t, e^{-t} gives

$$y_v = v \sinh t + \int_0^t \sinh(t-s)f(s) ds.$$

Since $0 < f \leq 1$, $0 < I \leq \int_0^\infty e^{-s} ds = 1$, with strict inequality $I < 1$.

Step 2: Identify the necessary slope. Expanding the hyperbolic sine yields

$$e^{-t}y_v = \frac{v}{2}(1 - e^{-2t}) + \frac{1}{2} \int_0^t e^{-s} f(s) ds - \frac{1}{2} e^{-2t} \int_0^t e^s f(s) ds.$$

The last term has magnitude at most $(e^{-t} - e^{-2t})/2$. Thus $e^{-t}y_v \rightarrow (v + I)/2$. Boundedness requires $\boxed{v_* = -I}$.

Step 3: Prove sufficiency and sign. Splitting the integral at $s = t$ and substituting $v = -I$ gives

$$y_*(t) = - \int_0^\infty K(t,s)f(s) ds, \quad K(t,s) = \begin{cases} e^{-t} \sinh s, & s \leq t, \\ e^{-s} \sinh t, & s \geq t. \end{cases}$$

This identity follows by grouping the e^t and e^{-t} terms above. Moreover,

$$\int_0^\infty K(t,s) ds = e^{-t}(\cosh t - 1) + e^{-t} \sinh t = 1 - e^{-t}.$$

Therefore $|y_*| \leq 1 - e^{-t} < 1$. For $t > 0$, the kernel and forcing are positive for $s > 0$, so $y_*(t) < 0$.

Step 4: Prove decay and sensitivity. Given $\varepsilon > 0$, choose A with $f(s) \leq \varepsilon$ for $s \geq A$. For $t > A$, the integral over $[0, A]$ has magnitude at most $e^{-t} \int_0^A \sinh s ds \rightarrow 0$; the remaining magnitude is at most $\varepsilon \int_A^\infty K \leq \varepsilon$. Hence $y_* \rightarrow 0$. If $v = v_* + \delta$, then $y_v = y_* + \delta \sinh t$, so $y_v/e^t \rightarrow \delta/2$. It tends to $+\infty$ for $\delta > 0$ and $-\infty$ for $\delta < 0$.

Question 8

For $t \geq 0$, solve $y'' + y = g(t)$ with zero initial value and slope. First use the continuous finite-duration forcing

$$g_2(t) = \begin{cases} \sin 2t, & 0 \leq t \leq \pi, \\ 0, & t > \pi. \end{cases}$$

Later compare it with g_3 , defined in the same way using $\sin 3t$.

Tasks

1. Derive the integral response and solve explicitly for g_2 on $[0, \pi]$.
2. Find the response after π , checking the matching value and slope. Does turning off the forcing return the solution to rest?
3. For any continuous forcing supported in $[0, \pi]$, derive the two integral conditions equivalent to $y(t) = 0$ for every $t \geq \pi$.
4. Test g_3 against these conditions and find its response. Explain how a nonzero forcing can leave no later motion.

Question 8 – Solution

Strategy. After the forcing ends, its two weighted integrals become the coefficients of the remaining homogeneous oscillation.

Step 1: Solve while the first forcing acts. Variation of parameters gives $y(t) = \int_0^t \sin(t-s)g(s) ds$. For g_2 on $[0, \pi]$, evaluation gives

$$y_2(t) = \frac{2 \sin t - \sin 2t}{3}.$$

Its value and slope at zero vanish, and $y_2'' + y_2 = \sin 2t$, independently verifying the integral evaluation.

Step 2: Match at the shutoff time. At π , $y_2 = 0$ and $y_2' = -4/3$. The subsequent homogeneous response is

$$y_2(t) = -\frac{4}{3} \sin(t - \pi) = \frac{4}{3} \sin t, \quad t \geq \pi.$$

It matches both data. Since $g_2(\pi) = 0$, $y'' = g_2 - y$ also matches. The resulting function is a classical C^2 solution, but it does not return to rest.

Step 3: Derive the cancellation conditions. For any such forcing and $t \geq \pi$,

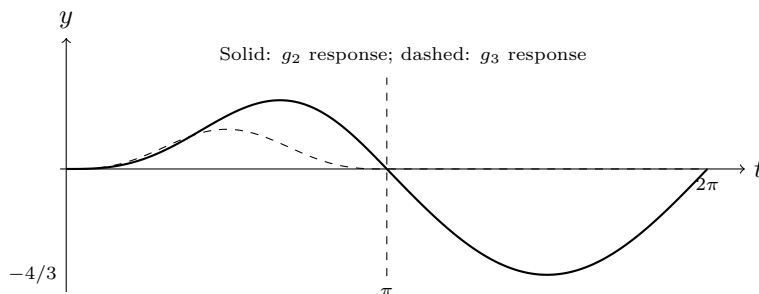
$$y(t) = \sin t \int_0^\pi \cos s g(s) ds - \cos t \int_0^\pi \sin s g(s) ds.$$

Independence of sine and cosine shows that the response vanishes identically after π exactly when both displayed integrals are zero. Cancellation of just one moment is insufficient.

Step 4: Construct a return to rest. For g_3 , product-to-sum identities give $\int_0^\pi \cos s \sin 3s ds = 0$ and $\int_0^\pi \sin s \sin 3s ds = 0$. Explicitly,

$$y_3(t) = \begin{cases} (3 \sin t - \sin 3t)/8, & 0 \leq t \leq \pi, \\ 0, & t \geq \pi. \end{cases}$$

On the first interval it has residual $\sin 3t$ and zero initial data; at π its value and slope are zero. The forcing's signed contributions cancel both final-state components, despite a nonzero response during the forcing interval.



Question 9

Let f be continuous on $[0, \pi]$. Seek C^2 solutions of

$$y'' + y = f(t), \quad y(0) = y(\pi) = 0.$$

Use variation of parameters; no general boundary-value theory is required.

Tasks

1. Write every solution satisfying the first endpoint condition, using its free initial slope A .
2. Derive a necessary and sufficient condition on f for the second endpoint condition. Explain uniqueness or nonuniqueness when the condition holds.
3. Apply the criterion to $f = 1$ and $f = \cos t$, and find all solutions in the latter case.
4. For $f = \cos t$, impose the additional condition $\int_0^\pi y(t) dt = 0$. Determine whether it selects a unique solution and find it.

Question 9 – Solution

Strategy. Variation of parameters supplies all initial-value responses; evaluating the integral at the second endpoint reveals the obstruction.

Step 1: Keep the free slope. The homogeneous pair is $\cos t, \sin t$, with $W = 1$. Integrating the parameter derivatives from zero and imposing $y(0) = 0$ gives every candidate:

$$y(t) = A \sin t + \int_0^t \sin(t-s)f(s) ds, \quad y'(0) = A.$$

Differentiating this formula verifies the equation and shows no candidate has been lost.

Step 2: Evaluate the second endpoint. Since $\sin \pi = 0$ and $\sin(\pi - s) = \sin s$, the remaining condition is

$$\int_0^\pi \sin s f(s) ds = 0.$$

If it fails, no choice of A helps. If it holds, every real A works: there are infinitely many solutions, differing by $A \sin t$. Thus this endpoint pair never selects exactly one solution for a compatible forcing.

Step 3: Compare the two forcings. For $f = 1$, the integral is 2, so there is no solution. For $f = \cos t$, it is $\int_0^\pi \sin s \cos s ds = 0$. Evaluation of the response integral, or a direct residual check, gives

$$y(t) = (A + t/2) \sin t, \quad A \in \mathbb{R}.$$

Indeed, $(t \sin t/2)'' + t \sin t/2 = \cos t$, and both endpoints vanish.

Step 4: Apply the extra measurement. The needed integrals are $\int_0^\pi \sin t dt = 2$ and, by integration by parts, $\int_0^\pi t \sin t dt = \pi$. Thus

$$0 = 2A + \pi/2, \quad A = -\pi/4, \quad y(t) = (t/2 - \pi/4) \sin t.$$

The coefficient of A in this measurement is nonzero, so it selects exactly one member of the compatible family. This does not contradict the earlier nonuniqueness: the added integral condition supplies new information.

Question 10

For $t \geq 0$, solve

$$y'' + y = e^{-t^2}, \quad y(0) = y'(0) = 0.$$

Define the convergent integrals $C = \int_0^\infty e^{-s^2} \cos s \, ds$ and $S = \int_0^\infty e^{-s^2} \sin s \, ds$. No special-function evaluations of C or S are required.

Tasks

1. Use variation of parameters to express the exact solution through two finite definite integrals, and verify its initial data and residual.
2. Show that the solution approaches the periodic function $P(t) = C \sin t - S \cos t$ in the sense that $y(t) - P(t) \rightarrow 0$.
3. Prove the quantitative estimate $|y(t) - P(t)| \leq e^{-t^2}/(2t)$ for every $t > 0$.
4. Prove $S > 0$ by pairing successive positive and negative half-waves of sine. Use this to show that y does not tend to zero, even though the forcing does.

Question 10 – Solution

Strategy. Keep the variation-of-parameters integrals exact, then bound their tails to identify the surviving oscillation.

Step 1: Write and check the exact response. With $W = 1$, the parameter derivatives are $u'_1 = -e^{-t^2} \sin t$ and $u'_2 = e^{-t^2} \cos t$. Hence

$$y(t) = \sin t \int_0^t e^{-s^2} \cos s \, ds - \cos t \int_0^t e^{-s^2} \sin s \, ds.$$

Equivalently, $y = \int_0^t \sin(t-s)e^{-s^2} \, ds$. Differentiating gives $y' = \int_0^t \cos(t-s)e^{-s^2} \, ds$ and $y'' = e^{-t^2} - y$; both initial data vanish.

Step 2: Isolate the tails. Absolute convergence follows from the integrability of e^{-s^2} (for $s \geq 1$, it is at most e^{-s}). Subtracting the full integrals gives

$$y(t) - P(t) = - \int_t^\infty \sin(t-s)e^{-s^2} \, ds.$$

Thus $|y - P| \leq \int_t^\infty e^{-s^2} \, ds \rightarrow 0$.

Step 3: Obtain an explicit error bound. For $s \geq t > 0$, $1 \leq s/t$. Consequently

$$|y - P| \leq \frac{1}{t} \int_t^\infty s e^{-s^2} \, ds = \boxed{\frac{e^{-t^2}}{2t}}.$$

Combining the tails before estimating avoids an unnecessary sum of two bounds.

Step 4: Prove a nonzero surviving oscillation. Absolute convergence permits pairing intervals. Their k th pair contributes

$$\int_0^\pi \sin u [e^{-(2k\pi+u)^2} - e^{-((2k+1)\pi+u)^2}] \, du > 0, \quad k \geq 0.$$

Therefore $S > 0$. Along $t = 2n\pi$, $y(t) \rightarrow -S$; along $t = (2n+1)\pi$, $y(t) \rightarrow S$. The solution has no limit and does not decay to zero. A decaying forcing can leave a persistent homogeneous oscillation.

