

Question 1

Consider the following differential equations for a real function $y(t)$:

$$\begin{aligned} \text{A : } (ty')' + y &= \sin t, & \text{B : } y'' + ty' + y^2 &= 0, \\ \text{C : } y'' + (y')^2 &= 0, & \text{D : } y'' + t^2y &= 0. \end{aligned}$$

An equation is linear in y if y and its derivatives occur to the first power, are not multiplied together, and have coefficients depending only on t . The terms homogeneous and nonhomogeneous here refer to linear equations with zero and nonzero forcing, respectively.

Tasks

1. Expand equation A. Classify all four equations by order and linearity; classify the linear ones by homogeneity.
2. Explain why t^2y does not destroy linearity, whereas y^2 and $(y')^2$ do. Does a zero right-hand side make B or C a homogeneous linear equation?
3. Put A in the form $y'' + p(t)y' + q(t)y = g(t)$. Find the largest open interval containing $t = 1$ on which these coefficients are continuous.
4. Suppose a C^2 solution of the undivided equation A passes through $t = 0$. Derive the necessary relation between $y(0)$ and $y'(0)$. Can $y(0) = 1$, $y'(0) = 0$ be prescribed there?

Question 1 – Solution

Strategy. Expand derivatives before classifying, and distinguish an equation from its normalized form at a zero leading coefficient.

Step 1: Expand and classify. The product rule gives $(ty')' = ty'' + y'$. Hence A is

$$ty'' + y' + y = \sin t.$$

The classification is

equation	order	type
A	2	linear, nonhomogeneous
B	2	nonlinear
C	2	nonlinear
D	2	linear, homogeneous

A is a second-order equation whose leading coefficient vanishes at one point; its order as an equation is not reclassified solely at that point.

Step 2: Identify what is being squared. In D, t^2 is a known coefficient, while y occurs to the first power. In B and C the unknown function or its derivative is squared. A zero right-hand side does not remove that nonlinearity. Under the stated convention, neither B nor C is a homogeneous linear equation.

Step 3: Normalize on a regular interval. For $t \neq 0$, division by t yields

$$y'' + \frac{1}{t}y' + \frac{1}{t}y = \frac{\sin t}{t}.$$

All three normalized coefficients are continuous on $(0, \infty)$, the largest open interval containing 1 with this property. Although $\sin t/t$ has a removable singularity at zero, $1/t$ does not.

Step 4: Test the original equation at zero. For a C^2 solution, the undivided equation at zero requires

$$0 \cdot y''(0) + y'(0) + y(0) = 0, \quad y'(0) + y(0) = 0.$$

The proposed data give $0 + 1 \neq 0$, so no such solution exists. The condition is necessary; deriving it alone is not an existence or uniqueness proof for every compatible pair. Division by the leading coefficient must not hide this compatibility requirement.

Question 2

An unknown trajectory satisfies

$$y'' = 6t, \quad y(1) = 2, \quad y'(1) = -1.$$

A student writes $y = t^3 + C$ as the general solution because differentiating twice gives $6t$.

Tasks

1. Find every C^2 solution of the differential equation by integrating twice. Explain the role of each integration constant.
2. Apply both initial conditions and verify the resulting solution in the equation and the data.
3. Diagnose the student's proposed general solution. Which initial slopes at $t = 1$ can that smaller family realize?
4. Retain only $y(1) = 2$. Describe all resulting solutions and their slopes at $t = 1$. Prove directly that restoring the slope condition selects exactly one solution.

Question 2 – Solution

Strategy. Integrate each derivative level separately, then examine which independent data the constants control.

Step 1: Recover the full family. One integration gives $y' = 3t^2 + A$; another gives

$$\boxed{y = t^3 + At + B}, \quad A, B \in \mathbb{R}.$$

These are all solutions on any interval: if two functions have second derivative $6t$, their difference has zero second derivative and therefore is affine. The constant A changes velocity; B changes position without changing velocity.

Step 2: Fit and verify both data. At $t = 1$, the conditions become $1 + A + B = 2$ and $3 + A = -1$. Thus $A = -4$, $B = 5$, giving

$$\boxed{y = t^3 - 4t + 5}.$$

Indeed, $y' = 3t^2 - 4$, $y'' = 6t$, $y(1) = 2$, and $y'(1) = -1$. The polynomial is defined on all of $\mathbb{R}$.

Step 3: Locate the missing freedom. Every function $t^3 + C$ solves the differential equation, but every one has derivative $3t^2$. In particular, its slope at $t = 1$ is always 3. It cannot realize the required slope -1 , regardless of C . Verification of a family does not by itself prove that the family is general.

Step 4: Separate position from velocity. With only $y(1) = 2$, we obtain $B = 1 - A$, so all possibilities are

$$\boxed{y = t^3 + A(t - 1) + 1}, \quad y'(1) = 3 + A.$$

The initial slope ranges over every real number as A varies. Setting it to -1 forces $A = -4$ and hence the single solution above.

Alternatively, the difference w of any two solutions with both prescribed data satisfies $w'' = 0$, $w(1) = w'(1) = 0$. Writing $w = at + b$ gives $a = 0$ and $b = 0$. This proves uniqueness directly, rather than merely counting constants.

Question 3

The functions e^t and e^{-t} are supplied as candidate solutions of

$$y'' - y = 0.$$

You may use this theorem: if p, q, g are continuous on an open interval I , then $y'' + py' + qy = g$ with $y(t_0) = a$, $y'(t_0) = b$, $t_0 \in I$, has exactly one solution on I .

Tasks

1. Verify that every $y = Ae^t + Be^{-t}$ solves the equation, without using a characteristic equation.
2. Determine A, B when $y(\ln 2) = 5/2$ and $y'(\ln 2) = 3/2$. Verify both data.
3. For arbitrary real data $y(\ln 2) = a$, $y'(\ln 2) = b$, solve for A, B . Show that every pair of initial data can be realized.
4. Use the theorem to prove that the displayed family contains every solution on $\mathbb{R}$. Explain why having two named constants would not be sufficient if the proposed family were $Ce^t + De^t$ instead.

Question 3 – Solution

Strategy. Check the functions, solve the two data equations, then use uniqueness to establish completeness.

Step 1: Verify the entire family. Differentiation gives

$$y' = Ae^t - Be^{-t}, \quad y'' = Ae^t + Be^{-t} = y.$$

Therefore $y'' - y = 0$ for every A, B . No root-solving method is required.

Step 2: Fit the stated data at their actual base point. Since $e^{\ln 2} = 2$ and $e^{-\ln 2} = 1/2$, the equations are

$$2A + B/2 = 5/2, \quad 2A - B/2 = 3/2.$$

Adding and subtracting yields $A = 1$, $B = 1$. Thus

$$\boxed{y = e^t + e^{-t}}.$$

Its value at $\ln 2$ is $2 + 1/2 = 5/2$, and its derivative there is $2 - 1/2 = 3/2$.

Step 3: Solve the general data problem. For arbitrary a, b the same equations give

$$4A = a + b, \quad B = a - b, \quad \boxed{A = (a + b)/4, \quad B = a - b}.$$

There is exactly one coefficient pair for each $(a, b) \in \mathbb{R}^2$. Substitution returns $y(\ln 2) = a$ and $y'(\ln 2) = b$, so the claim includes all data, not only the numerical example.

Step 4: Establish completeness, not just verification. Here $p = 0$, $q = -1$, $g = 0$ are continuous on $\mathbb{R}$. Any solution has some data a, b at $\ln 2$. Step 3 constructs a member of the displayed family with those same data; uniqueness forces agreement on $\mathbb{R}$.

By contrast, $Ce^t + De^t = (C + D)e^t$ has only one effective parameter. Every such function satisfies $y' = y$, so at the base point it can realize only data with $b = a$. Two symbols for constants do not necessarily provide two independent freedoms.

Question 4

For $y'' + y = 0$, you may use the complete family

$$y(t) = A \cos t + B \sin t.$$

Consider two endpoint conditions $y(0) = 0$ and $y(L) = c$, where $L > 0$ is specified. Compare these with initial data $y(0) = 0$, $y'(0) = b$.

Tasks

1. Solve the initial-data problem for an arbitrary real b . Explain why it always determines exactly one member of the family.
2. Determine all solutions of the endpoint problem for $L = \pi/2$ and arbitrary c .
3. Determine all solutions for $L = \pi$, treating $c = 0$ and $c \neq 0$ separately. Explain why two conditions need not select one solution.
4. For $L = \pi$, $c = 0$, sketch the solutions with $B = -1, 0, 1, 2$. Explain why their common endpoint values do not violate uniqueness for initial-value problems.

Question 4 – Solution

Strategy. Translate each condition into a constraint on A, B and check whether the second constraint supplies new information.

Step 1: Use value and slope at the same point. From $y(0) = A$ and $y'(0) = B$, initial data $0, b$ give

$$y = b \sin t.$$

Both constants are fixed, for every real b . The coefficients of the normalized equation are continuous everywhere, consistent with initial-value uniqueness.

Step 2: Move the second datum to $\pi/2$. The first condition gives $A = 0$. At the other endpoint,

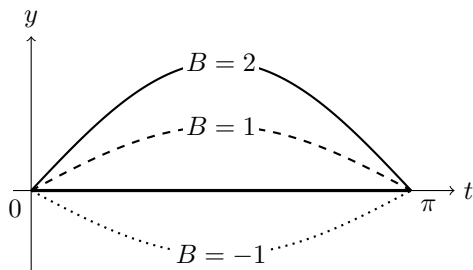
$$y(\pi/2) = B = c, \quad y = c \sin t.$$

This particular pair of endpoint conditions also determines one solution.

Step 3: Detect a redundant or incompatible condition. At $L = \pi$, every function $B \sin t$ has $y(\pi) = 0$. Consequently,

$$c = 0 : y = B \sin t \ (B \in \mathbb{R}), \quad c \neq 0 : \text{no solution}.$$

When $c = 0$, the second condition adds no restriction on B . When $c \neq 0$, it contradicts the first condition and the equation. Merely counting two stated conditions misses this distinction.



The horizontal solution is $B = 0$.

Step 4: Compare endpoint agreement with initial-data agreement. The four curves share their endpoint positions but have different initial slopes B . At $t = \pi$ their slopes are $-B$, also different. The initial-value theorem requires agreement of both value and first derivative at the same point. Agreement of values at two separate points is a different requirement, so the sketch creates no contradiction.

Question 5

For a normalized linear equation $y'' + p(t)y' + q(t)y = g(t)$, continuity of p, q, g on an open interval containing the initial time guarantees a unique solution throughout that interval for any two initial data.

Consider

$$t(t-2)y'' + (t+1)y' + \ln(t+3)y = \frac{1}{t-4}.$$

All functions and solutions are real.

Tasks

1. Find the domain of the displayed coefficients before normalization. List the additional points excluded when solving for y'' .
2. Find the largest theorem-guaranteed interval for initial time $t_0 = 1$, and then for $t_0 = 3$, with arbitrary finite initial value and slope.
3. Does a finite endpoint of such an interval prove that every solution blows up there? Explain precisely what the theorem does and does not say.
4. Test that distinction on the separate equation $ty'' + y' = 0$ with data $y(1) = a$, $y'(1) = b$. Solve on $(0, \infty)$ and determine exactly which resulting solutions extend as C^2 solutions of the undivided equation across zero.

Question 5 – Solution

Strategy. Locate coefficient obstructions before applying the theorem, then compare its guaranteed interval with the behavior of actual solutions.

Step 1: Separate domain from degeneracy. The logarithm requires $t > -3$ and the forcing excludes $t = 4$. Thus the original coefficient domain is $(-3, 4) \cup (4, \infty)$. Normalization additionally excludes $t = 0, 2$, where the leading coefficient vanishes. For other allowed t ,

$$p = \frac{t+1}{t(t-2)}, \quad q = \frac{\ln(t+3)}{t(t-2)}, \quad g = \frac{1}{(t-4)t(t-2)}.$$

Step 2: Choose the connected interval containing the data. The regular intervals are $(-3, 0)$, $(0, 2)$, $(2, 4)$ and $(4, \infty)$. Therefore the largest applicable intervals are

$$\boxed{t_0 = 1 : (0, 2)}, \quad \boxed{t_0 = 3 : (2, 4)}.$$

The initial values do not change these coefficient-based guarantees.

Step 3: State the limitation accurately. The theorem guarantees existence and uniqueness within the selected interval. It does not classify the behavior at a singular endpoint or prove that every solution becomes unbounded there. Whether a particular solution extends requires examination of that solution and the original equation. A coefficient-domain boundary is also distinct from a pole in the solution itself.

Step 4: Solve the comparison equation completely. Since $(ty')' = ty'' + y'$, integration on $t > 0$ gives $ty' = C$, then $y = C \ln t + D$. The data at 1 imply $C = b$, $D = a$, so

$$\boxed{y = a + b \ln t \quad (t > 0)}.$$

If $b \neq 0$, y is unbounded as $t \downarrow 0$, so no continuous, hence no C^2 , extension exists. If $b = 0$, the constant solution $y = a$ satisfies the undivided equation on all of $\mathbb{R}$.

Thus extension occurs exactly when $\boxed{b = 0}$. In fact it is unique among C^2 extensions: on the negative side every solution has form $C_- \ln |t| + D_-$; continuity forces $C_- = 0$, $D_- = a$. The normalized theorem alone did not provide that endpoint conclusion.

Question 6

Let $L[y] = y'' + y$ and consider the forced equation $L[y] = t$. The functions t , $\sin t$ and $\cos t$ are supplied for direct verification.

Tasks

1. Compute $L[t]$, $L[\sin t]$ and $L[\cos t]$. Determine exactly which constants a, b, c make $at + b\sin t + c\cos t$ solve the forced equation on $\mathbb{R}$.
2. Let $u = t + \sin t$ and $v = t + \cos t$. Determine the equations satisfied by $u + v$, $u - v$ and $(u + v)/2$.
3. For arbitrary solutions u, v of $L[y] = t$, determine exactly which real weights α, β ensure $\alpha u + \beta v$ is again a solution. Explain the difference from homogeneous superposition.
4. Find a solution with $y(0) = 2$, $y'(0) = 0$. Verify it and justify uniqueness using the continuous-coefficient initial-value theorem.

Question 6 – Solution

Strategy. Apply the linear operator to the proposed combination and track the forcing as carefully as the unknown function.

Step 1: Identify the forced and zero-forcing pieces. Direct differentiation yields

$$L[t] = t, \quad L[\sin t] = 0, \quad L[\cos t] = 0.$$

Linearity then gives $L[at + b\sin t + c\cos t] = at$. Equality with t for every real t requires $a = 1$; b, c are arbitrary. Hence all such candidates are

$$\boxed{y = t + b\sin t + c\cos t}.$$

The forcing fixes the coefficient of t , leaving the two homogeneous freedoms.

Step 2: Track sums, differences and averages. Because $L[u] = L[v] = t$,

$$L[u + v] = 2t, \quad L[u - v] = 0, \quad L[(u + v)/2] = t.$$

The sum is generally not a solution of the original forced equation. The difference solves the associated homogeneous equation, and the average preserves the original forcing.

Step 3: Determine the allowed weights. For any such u, v ,

$$L[\alpha u + \beta v] = (\alpha + \beta)t.$$

This equals t on $\mathbb{R}$ exactly when $\boxed{\alpha + \beta = 1}$. The weights need not be nonnegative. For homogeneous solutions, by contrast, the right-hand side remains zero for every pair of weights. Also, the zero function is not a solution of $L[y] = t$; the forced solution set does not have all the closure properties of the homogeneous one.

Step 4: Fit and verify the initial data. For the candidate family, $y(0) = c$ and $y'(0) = 1 + b$. The data give $c = 2$, $b = -1$, so

$$\boxed{y = t - \sin t + 2\cos t}.$$

Its derivative is $1 - \cos t - 2\sin t$, its initial value and slope are 2 and 0, and $y'' + y = t$. Since $p = 0$, $q = 1$, $g = t$ are continuous on $\mathbb{R}$, the initial-value theorem proves uniqueness there. No general method for finding particular solutions was assumed; the necessary candidate was supplied.

Question 7

Consider $y'' + 4y = 0$ with $y(0) = 1$, $y'(0) = 2$. Define

$$E(t) = (y'(t))^2 + 4(y(t))^2.$$

For direct verification you may use the candidate $y(t) = \cos(2t) + \sin(2t)$.

Tasks

1. Prove that E is constant for every solution and calculate its value for the stated data.
2. Deduce a bound on $|y(t)|$ without first substituting a solution formula. Verify the candidate and determine a time when the bound is attained.
3. In coordinates (y, v) with $v = y'$, sketch the curve forced by the energy identity, mark the initial state and indicate the direction of motion there.
4. Prove uniqueness directly: if two solutions share both initial data, apply the same identity to their difference. Explain why knowing the energy alone does not specify a unique solution.

Question 7 – Solution

Strategy. Differentiate a sum of nonnegative terms, then use its conservation both to bound a solution and to eliminate a zero-data difference.

Step 1: Derive the conserved quantity. For any solution,

$$E' = 2y'y'' + 8yy' = 2y'(y'' + 4y) = 0.$$

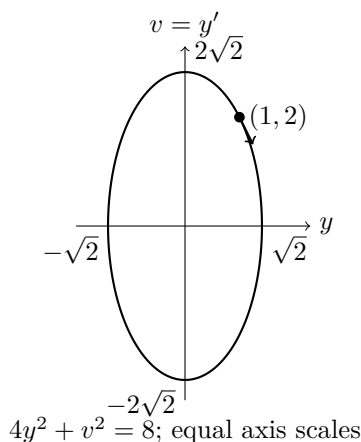
Thus $E(t) = E(0)$ throughout its interval. The prescribed data give

$$E = 2^2 + 4(1)^2 = 8.$$

Step 2: Obtain and attain the displacement bound. Since $(y')^2 \geq 0$, $4y^2 \leq 8$, whence $|y| \leq \sqrt{2}$. For the supplied candidate,

$$y' = -2\sin(2t) + 2\cos(2t), \quad y'' = -4y.$$

It has the required data, and at $t = \pi/8$ it has $y = \sqrt{2}$, $y' = 0$. Therefore the bound is sharp, not just a convenient overestimate.



Step 3: Interpret the state curve. Writing $v = y'$ gives $4y^2 + v^2 = 8$, an ellipse with intercepts $y = \pm\sqrt{2}$ and $v = \pm 2\sqrt{2}$. The initial point is $(1, 2)$. Its velocity in these coordinates is

$$(y', v') = (v, -4y) = (2, -4),$$

so it moves rightward and downward there. The axes represent state variables, not time versus displacement.

Step 4: Prove uniqueness and distinguish energy from data. The difference w of two solutions with the same initial data satisfies $w'' + 4w = 0$ and $w(0) = w'(0) = 0$. Its conserved quantity is $(w')^2 + 4w^2 = 0$. Both terms are nonnegative, so $w = 0$ everywhere on their common interval.

Energy alone only fixes an ellipse. For example, $\sqrt{2}\cos(2t)$ and $\sqrt{2}\sin(2t)$ both have energy 8 but different initial states. Initial position and velocity identify a point and its motion; the scalar energy value does not replace them.

Question 8

Suppose both $y_1(t) = 1$ and $y_2(t) = t^2$ solve an unknown normalized homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

on $(0, \infty)$.

Tasks

1. Determine p and q directly from the two supplied solutions. Check your result by substitution.
2. Can any normalized equation of this form, with continuous p, q on an interval containing zero, have both supplied functions as solutions there? Prove your answer without a determinant formula.
3. Multiply the equation from Task 1 by t . Show that the undivided equation admits $y = C_1 + C_2t^2$ on all of $\mathbb{R}$. Determine all members of this family satisfying $y(0) = a$, $y'(0) = b$.
4. Sketch $y = t^2$ and $y = 2t^2$ near zero. Explain how distinct solutions can share the same tangent there without contradicting the regular linear initial-value theorem.

Question 8 – Solution

Strategy. Insert simple known solutions to recover coefficients, then inspect what changes when the leading coefficient vanishes.

Step 1: Recover the equation on its stated interval. Inserting $y_1 = 1$ gives $q(t) = 0$. Inserting $y_2 = t^2$ then gives $2 + 2tp(t) = 0$, so

$$\boxed{p(t) = -1/t, \quad q(t) = 0}, \quad \boxed{y'' - \frac{1}{t}y' = 0 \quad (t > 0)}.$$

The constant solution has both derivatives zero, and $2 - (1/t)(2t) = 0$ verifies the quadratic solution.

Step 2: Rule out a regular equation through zero. At $t = 0$, the function t^2 has value 0, first derivative 0 and second derivative 2. In any normalized equation with finite coefficients at zero it would require

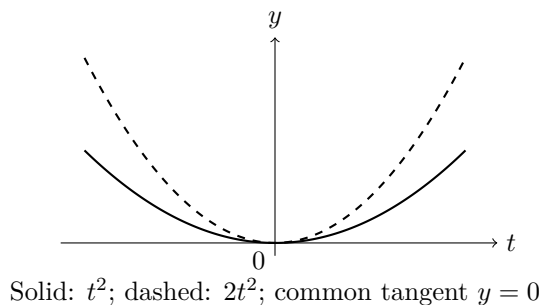
$$2 + p(0) \cdot 0 + q(0) \cdot 0 = 0,$$

which is impossible. Thus no continuous-coefficient normalized equation on an interval containing zero can have both supplied functions as solutions. The obstruction is already visible from the quadratic alone.

Step 3: Retain the degenerate leading coefficient. The undivided equation is $ty'' - y' = 0$. For $y = C_1 + C_2t^2$, its left-hand side is $t(2C_2) - 2C_2t = 0$, including at zero. The family has $y(0) = C_1$, $y'(0) = 0$, so

$$\boxed{b = 0 : y = a + C_2t^2 \ (C_2 \in \mathbb{R})}, \quad \boxed{b \neq 0 : \text{no member}}.$$

Indeed, the undivided equation itself at zero forces $y'(0) = 0$, so nonzero b permits no C^2 solution at all.



Step 4: Explain the shared tangent. The two plotted functions share value 0 and slope 0 at zero but have different curvatures. The coefficient of y'' vanishes there, so the equation does not determine the second derivative from those data. Its normalized coefficient $-1/t$ is not continuous at the initial point. The regular linear theorem therefore does not apply, and the multiple zero-data solutions do not contradict it.

Question 9

Consider the nonlinear equation

$$y'' = \sqrt{|y|}, \quad y(0) = 0, \quad y'(0) = 0.$$

For a real parameter $a \geq 0$, define

$$y_a(t) = \begin{cases} 0, & t \leq a, \\ (t-a)^4/144, & t > a. \end{cases}$$

A classical solution here means a real C^2 function satisfying the equation at every point of its interval.

Tasks

1. Verify the equation on each open side of $t = a$.
2. Check the value, first derivative and second derivative at the joining point. Does the equation hold there as well?
3. Verify both initial conditions for every $a \geq 0$, and show that different finite choices of a give different solutions. Identify one additional solution.
4. Sketch y_0 , y_1 and the zero solution on $[0, 3]$. Explain why continuity of the right-hand side does not justify applying the linear initial-value uniqueness theorem to this example. Does an assigned position and slope always determine a unique trajectory?

Question 9 – Solution

Strategy. Verify the join at the derivative order required by the equation, then use the family as a counterexample to an overgeneralized uniqueness claim.

Step 1: Verify away from the join. For $t < a$, both sides of the equation are zero. For $t > a$,

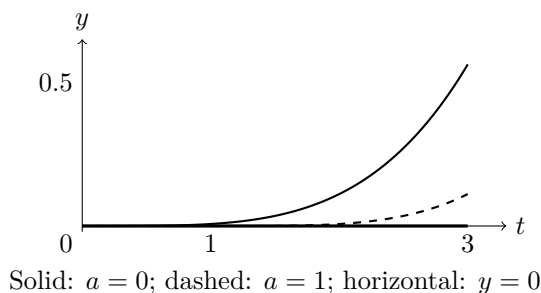
$$y'_a = \frac{(t-a)^3}{36}, \quad y''_a = \frac{(t-a)^2}{12}, \quad \sqrt{|y_a|} = \sqrt{\frac{(t-a)^4}{144}} = \frac{(t-a)^2}{12}.$$

The coefficient 144 is essential to this identity.

Step 2: Verify classical regularity at the join. The right-hand limits of y_a, y'_a, y''_a as $t \downarrow a$ are all zero, matching the values on the left. The first and second derivatives at the join are also zero, as follows directly from their difference quotients. Hence y_a is C^2 on $\mathbb{R}$.

At $t = a$, the equation reads $0 = \sqrt{|0|}$, so it holds there, not merely away from the join. There is no hidden corner in position or velocity.

Step 3: Check the common data and distinctness. Since $a \geq 0$, every family member has $y_a(0) = y'_a(0) = 0$, including $a = 0$ by the join calculation. If $a_1 < a_2$, choose $a_1 < t < a_2$. Then $y_{a_1}(t) > 0$ but $y_{a_2}(t) = 0$, proving they are different solutions. The function $\boxed{y \equiv 0}$ is another solution, different from every finite-delay member.



Step 4: State exactly what has failed. The right-hand side $\sqrt{|y|}$ is continuous but depends nonlinearly on the unknown function. This is not an equation $y'' + p(t)y' + q(t)y = g(t)$ with prescribed continuous coefficients, so that linear theorem cannot be invoked. The verified family proves that continuity alone does not ensure uniqueness for a general nonlinear second-order equation.

Two initial data determine one solution under appropriate hypotheses, not for every differential equation. This example exhibits infinitely many solutions; it does not claim to classify all solutions of the nonlinear equation.

Question 10

A prescribed acceleration changes abruptly at $t = 1$:

$$y'' = \begin{cases} 0, & 0 \leq t < 1, \\ 2, & t > 1, \end{cases} \quad y(0) = 0, \quad y'(0) = 1.$$

Seek a function that is C^1 on $[0, \infty)$ and C^2 on each open side of 1, satisfying the equation there. No value for y'' at 1 is assumed. Thus position and velocity must be continuous at the switch.

Tasks

1. Integrate on the first phase and find the position and velocity arriving at $t = 1$.
2. Integrate on the second phase with two new constants, determine them using the matching conditions, and give the complete piecewise trajectory.
3. Compute the one-sided derivatives of velocity at the switch. Can any assigned value of the forcing at $t = 1$ make this trajectory a classical C^2 solution through the switch?
4. Sketch position and velocity. A student adds $k(t - 1)$ to the second-phase position while leaving the first phase unchanged. Determine which k satisfy the stated requirements, and explain why position matching alone is insufficient.

Question 10 – Solution

Strategy. Carry both position and velocity across the switch, and distinguish a piecewise classical solution from one that is C^2 everywhere.

Step 1: Propagate the first phase. With zero acceleration and initial velocity 1, $y' = 1$ and $y = t$ for $0 \leq t \leq 1$. The arriving data are

$$\boxed{y(1) = 1, \quad y'(1) = 1}.$$

Step 2: Use both matching conditions. Writing the second-phase antiderivative about the switch gives

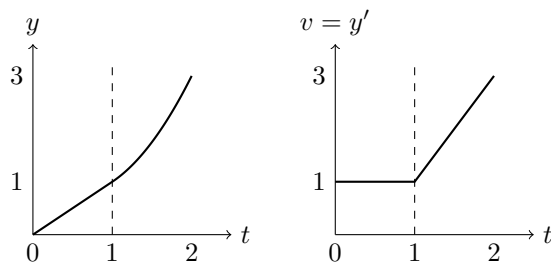
$$y = (t - 1)^2 + A(t - 1) + B \quad (t > 1).$$

Continuity of position sets $B = 1$; continuity of velocity sets $A = 1$. Thus

$$\boxed{y(t) = \begin{cases} t, & 0 \leq t \leq 1, \\ t + (t - 1)^2, & t > 1, \end{cases}} \quad y'(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 1 + 2(t - 1), & t > 1. \end{cases}$$

These formulas satisfy all the stated requirements and uniquely determine both phases.

Step 3: Check the actual regularity. The derivative of velocity at 1 has left-hand value 0 and right-hand value 2. Therefore $y''(1)$ does not exist. Assigning a forcing value at that single point cannot change the unequal one-sided limits. The trajectory is C^1 and piecewise C^2 , but is not a classical C^2 solution across the switch.



Position has a continuous tangent; velocity has a corner.

Step 4: Reject an unmatched velocity. Adding $k(t - 1)$ on $t > 1$ keeps the position continuous and does not change the second derivative on that open phase. However, the right-hand velocity at 1 becomes $1 + k$, while the left-hand velocity stays 1. Hence the C^1 requirement forces $\boxed{k = 0}$.

A second-order problem carries two pieces of state information. Position matching alone would admit a jump in velocity that is excluded by the problem's stated regularity; checking only the differential equation away from the switch would miss it.