

Question 1

Consider

$$y' = t - y, \quad y(0) = 1.$$

For a step h , explicit Euler uses $t_{n+1} = t_n + h$ and $Y_{n+1} = Y_n + hf(t_n, Y_n)$, where Y_n approximates $y(t_n)$. Use exact arithmetic in intermediate steps unless rounding is explicitly requested.

Tasks

1. Starting at $t_0 = 0$, use $h = 1/4$ to compute every Euler value through $t = 1$. Include the slope used on each step.
2. Find the exact solution and the signed endpoint error $Y_4 - y(1)$, reporting its numerical value to six decimal places.
3. Draw the Euler polygon and the exact solution on $[0, 1]$. Explain what the slope of each straight segment represents.
4. A program obtains $Y_1 = 0.8125$ by evaluating $t - y$ at (t_1, Y_0) . Identify the error and show the correct first update. Is that program implementing the stated Euler formula?

Question 1 – Solution

Strategy. Evaluate each slope at the current numerical point, then move both time and the approximation to the next node.

Step 1: Compute the four updates. The recurrence is $Y_{n+1} = Y_n + (t_n - Y_n)/4$. Its exact values are

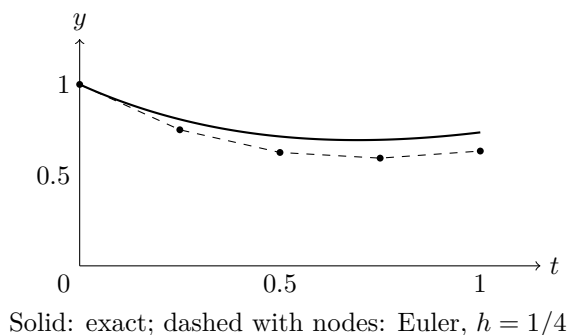
n	t_n	Y_n	$f(t_n, Y_n)$
0	0	1	-1
1	1/4	3/4	-1/2
2	1/2	5/8	-1/8
3	3/4	19/32	5/32
4	1	81/128	not needed

Thus $Y_4 = 81/128 = 0.6328125$.

Step 2: Check against the exact IVP. Solving $y' + y = t$ with integrating factor e^t gives

$$y = t - 1 + 2e^{-t}, \quad Y_4 - y(1) = \frac{81}{128} - \frac{2}{e} \approx -0.102946.$$

The exact formula has $y(0) = 1$ and derivative $1 - 2e^{-t} = t - y$, so it is the correct comparison solution.



Step 3: Interpret the polygon. On $[t_n, t_{n+1}]$, the Euler segment has slope $f(t_n, Y_n)$. It is tangent to the solution through the numerical point (t_n, Y_n) , which generally differs from the original exact solution after the first step. The polygon is continuous but usually has corners at its nodes.

Step 4: Diagnose the time index. The correct first update is $1 + (1/4)(0 - 1) = 0.75$. The program instead uses $1 + (1/4)(1/4 - 1) = 0.8125$, mixing the new time with the old state. It is not the stated explicit Euler method, which requires both arguments from the same current node.

Question 2

For $y' = y$, $y(0) = 1$, apply explicit Euler on $[0, 1]$ with N equal steps and $h = 1/N$. Let Y_N denote its endpoint value. You may use $\ln(1 + h) = h - h^2/2 + O(h^3)$ and $e^z = 1 + z + O(z^2)$ as the arguments tend to zero.

Tasks

1. Derive a closed formula for Y_N and prove that it underestimates the exact endpoint for every positive integer N .
2. Compute the one-step defect when Euler starts from an exact value $y(t_n)$. Show that this defect is of order h^2 .
3. Derive the leading endpoint error as $h \rightarrow 0$ and explain why its order differs from that of the one-step defect.
4. Calculate endpoint values and absolute errors for $N = 2, 4, 8$. Explain what error reduction is expected when N doubles and what the computation cost does.

Question 2 – Solution

Strategy. Solve the discrete recurrence exactly, then compare its logarithm with the continuous exponential growth.

Step 1: Find and bound the endpoint. Euler gives $Y_{n+1} = (1 + h)Y_n$, so

$$Y_N = (1 + 1/N)^N.$$

The exact solution is e^t . Since $\ln(1 + h) < h$ for $h > 0$, $N \ln(1 + 1/N) < 1$ and hence $Y_N < e$. This is an underestimate for every grid, not merely for the three grids below.

Step 2: Define the one-step defect. Starting with the exact value e^{t_n} , the difference between the exact next value and one Euler step is

$$d_n = e^{t_n+h} - e^{t_n}(1 + h) = e^{t_n}(e^h - 1 - h) = \frac{1}{2}e^{t_n}h^2 + O(h^3).$$

On $0 \leq t_n \leq 1$, this is uniformly $O(h^2)$. Here defect means an error in solution value; dividing it by h would use a different convention.

Step 3: Derive the accumulated error. Because $Nh = 1$,

$$\ln Y_N = \frac{\ln(1 + h)}{h} = 1 - \frac{h}{2} + O(h^2).$$

Exponentiating gives $Y_N = e(1 - h/2 + O(h^2))$, so

$$e - Y_N = \frac{e}{2}h + O(h^2).$$

There are $N = 1/h$ steps over the fixed interval, and their defects propagate through later updates. An $O(h^2)$ one-step defect therefore need not yield an $O(h^2)$ endpoint error; here the accumulated error is first order.

Step 4: Compare the grids.

N	Y_N	$e - Y_N$
2	2.250000	0.468282
4	2.441406	0.276876
8	2.565785	0.152497

The errors approach a factor-of-two reduction under each doubling of N , because their leading term is proportional to h . Exact halving is not promised on coarse grids. The number of slope evaluations doubles, exposing the first-order accuracy versus work tradeoff.

Question 3

Consider a decaying quantity satisfying

$$y' = -4y, \quad y(0) = 1.$$

Apply explicit Euler with a constant step $h > 0$ for arbitrarily many steps.

Tasks

1. Derive the numerical solution Y_n and determine exactly which steps give $Y_n \rightarrow 0$ as $n \rightarrow \infty$.
2. Determine the stricter step condition that keeps every numerical value nonnegative and the sequence nonincreasing. Discuss $h = 1/4$ and $h = 1/2$ separately.
3. Compute enough values for $h = 0.4$ and $h = 0.6$ to show their contrasting behavior, and compare them graphically with the exact decay.
4. Explain why decreasing the physical solution's magnitude does not by itself ensure decay of an Euler approximation, and distinguish stability from faithful positivity.

Question 3 – Solution

Strategy. Analyze the discrete amplification factor rather than transferring the differential equation's stability directly to its numerical update.

Step 1: Solve the recurrence. The update is $Y_{n+1} = (1 - 4h)Y_n$, so $Y_n = (1 - 4h)^n$. It tends to zero exactly when

$$|1 - 4h| < 1, \quad 0 < h < 1/2.$$

Step 2: Impose positivity and monotonicity. All values are nonnegative and nonincreasing exactly when $0 \leq 1 - 4h < 1$, or

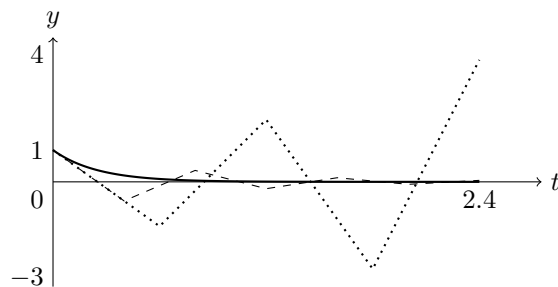
$$0 < h \leq 1/4.$$

At $h = 1/4$, the approximation becomes zero after one step, although e^{-4t} is strictly positive at every finite time. At $h = 1/2$, $Y_n = (-1)^n$ oscillates without decay. For $h > 1/2$, its magnitude grows without bound.

Step 3: Compare two oscillating sequences.

n	0	1	2	3	4
$h = 0.4$	1	-0.6	0.36	-0.216	0.1296
$h = 0.6$	1	-1.4	1.96	-2.744	3.8416

Each row is plotted at its own times $t_n = nh$, not at common node indices interpreted as time.



Solid: exact; dashed: $h = 0.4$; dotted: $h = 0.6$

Step 4: Interpret numerical stability. The exact solution $y = e^{-4t}$ decays smoothly. Euler replaces its per-step factor e^{-4h} by $1 - 4h$, which can be negative or have magnitude exceeding one. At $h = 0.4$ the method is asymptotically decaying but produces unphysical sign changes; at $h = 0.6$ it also loses decay. Stability controls long-run amplification, while positivity is an additional property with a stricter step requirement.

Question 4

A forced response obeys

$$y' = g(t) - y, \quad y(0) = 0, \quad g(t) = \begin{cases} 1, & 0 \leq t < 0.6, \\ 3, & t \geq 0.6. \end{cases}$$

The solution is continuous at the forcing switch; the differential equation is solved classically on each side. On any grid use $Y_{n+1} = Y_n + (t_{n+1} - t_n)[g(t_n) - Y_n]$.

Tasks

1. Compute Euler values on grid A: 0, 0.4, 0.8, 1.
2. Compute Euler values on grid B: 0, 0.4, 0.6, 1. Use the specified value of g at the switching node.
3. Derive the exact solution by matching the two phases and compare the two absolute endpoint errors.
4. Draw both Euler polygons and the exact solution. Explain why including the switch is important, yet does not guarantee the smaller endpoint error for these particular grids.

Question 4 – Solution

Strategy. Distinguish resolving an event from controlling the numerical error made within each phase.

Step 1: Compute grid A. The step lengths are 0.4, 0.4, 0.2. Left-node evaluation gives

$$Y(0.4) = 0.4, \quad Y(0.8) = 0.4 + 0.4(1 - 0.4) = 0.64, \quad \boxed{Y_A(1) = 0.64 + 0.2(3 - 0.64) = 1.112}.$$

The second step continues using $g = 1$ across an interval whose forcing actually changes partway through.

Step 2: Compute grid B. The step lengths are 0.4, 0.2, 0.4. Thus

$$Y(0.4) = 0.4, \quad Y(0.6) = 0.4 + 0.2(1 - 0.4) = 0.52, \quad \boxed{Y_B(1) = 0.52 + 0.4(3 - 0.52) = 1.512}.$$

At $t = 0.6$ the prescribed forcing is 3, so the final step uses the new phase.

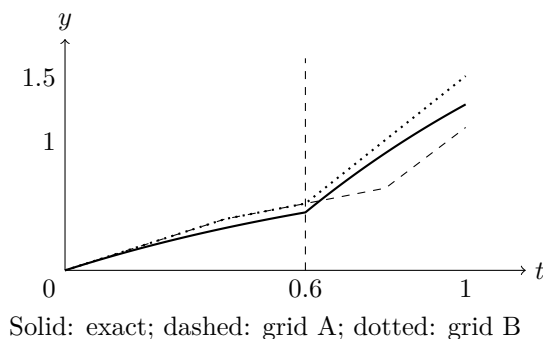
Step 3: Obtain the correct comparison value. The exact phase solutions are

$$y(t) = \begin{cases} 1 - e^{-t}, & 0 \leq t \leq 0.6, \\ 3 + (-2 - e^{-0.6})e^{-(t-0.6)}, & t \geq 0.6. \end{cases}$$

They agree at the switch and satisfy their respective equations. Consequently

$$y(1) = 3 - 2e^{-0.4} - e^{-1} \approx 1.291480,$$

so the absolute errors are $\boxed{0.179480 \text{ for A, } 0.220520 \text{ for B}}.$



Step 4: Explain the apparently better unaligned grid. Grid A undercounts the switched forcing, while grid B uses it correctly but takes a larger last step and overestimates the concave-down response. Here A happens to have the smaller endpoint error. Event alignment removes a known forcing mismatch, but does not order total errors on different coarse grids. Accurate computation also requires sufficiently small steps within the smooth phases.

Question 5

A normalized population satisfies

$$p' = 2p(1 - p), \quad 0 \leq p(0) \leq 1.$$

The exact solution preserves the physical interval $[0, 1]$. For explicit Euler with step $h > 0$, write $P_{n+1} = F_h(P_n)$. For the local discrete comparison, a small perturbation near $P = 1$ is multiplied to first order by $F'_h(1)$; call the linearized update contracting when $|F'_h(1)| < 1$.

Tasks

1. Derive F_h and find exactly which positive steps map all of $[0, 1]$ back into $[0, 1]$.
2. Prove the condition is necessary, not merely sufficient, by producing an overshooting initial value whenever it fails.
3. Compute the first two Euler updates for $h = 1$, $P_0 = 3/4$, and explain why they misrepresent the exact solution's qualitative behavior.
4. Find the step range for linearized contraction at 1. Compare it with interval preservation, and sketch the update maps for $h = 1/4$ and $h = 1$.

Question 5 – Solution

Strategy. Check the whole physical interval, not only the behavior of tiny perturbations near its upper endpoint.

Step 1: Factor the interval constraint. Euler gives $F_h(p) = p + 2hp(1-p)$. On $[0, 1]$ it is nonnegative. Also

$$1 - F_h(p) = (1-p)(1-2hp).$$

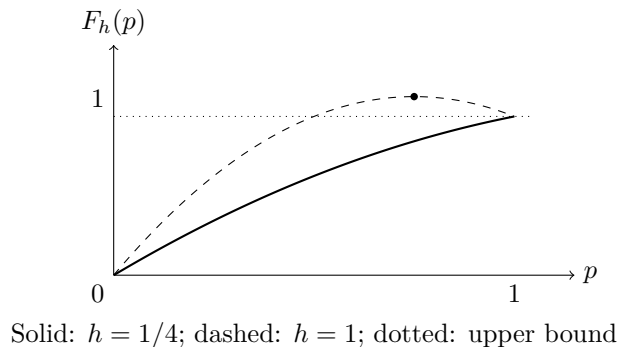
If $0 < h \leq 1/2$, both factors are nonnegative for every $p \in [0, 1]$, so $F_h(p) \leq 1$. Therefore $0 < h \leq 1/2$ preserves the interval at every step by induction.

Step 2: Prove necessity. If $h > 1/2$, choose p strictly between $1/(2h)$ and 1. Then $1-p > 0$ but $1-2hp < 0$, so $F_h(p) > 1$. Such a value exists for every $h > 1/2$, proving the stated condition is exact.

Step 3: Exhibit overshoot. For $h = 1$, $F_1(p) = 3p - 2p^2$, giving

$$P_1 = 9/8 = 1.125, \quad P_2 = 27/32 = 0.84375.$$

The numerical solution crosses above the capacity and then drops. The exact solution starting at $3/4$ increases toward 1 from below and cannot cross that equilibrium, by uniqueness.



Step 4: Distinguish two step restrictions. Since $F'_h(p) = 1 + 2h - 4hp$, $F'_h(1) = 1 - 2h$. Linearized contraction therefore holds for $0 < h < 1$, a larger range than interval preservation. At $h = 1$ the linearized magnitude equals one, so the strict contraction test makes no decision about nonlinear asymptotic behavior.

A locally contracting update can still move some valid states above the physical bound. The map criterion protects the entire interval; the derivative criterion only describes first-order behavior near one fixed point.

Question 6

Consider

$$y' = y^2, \quad y(0) = 1.$$

Two algorithms use a grid with $h = 1/4$. Algorithm A applies explicit Euler directly to y . Algorithm B first sets $u = 1/y$, applies explicit Euler to the transformed equation, and then recovers $y = 1/u$ wherever possible.

Tasks

1. Derive both update rules and compute their approximations at $t = 1/4$ and $t = 1/2$.
2. Find the exact IVP solution and determine which algorithm is exact at these nodes. Explain why this happens here.
3. Starting from a common positive value y with $hy < 1$, compare the two one-step updates algebraically and find their exact difference.
4. Explain what happens to algorithm B at $t = 1$ and why finite direct-Euler values at or beyond that time cannot establish continuation of the original IVP.

Question 6 – Solution

Strategy. A nonlinear change of dependent variable preserves the exact differential equation but generally does not commute with a finite Euler step.

Step 1: Derive and compute the algorithms. Direct Euler gives $Y_{n+1} = Y_n + hY_n^2$, $Y_0 = 1$. Since $u' = -y'/y^2 = -1$, transformed Euler gives $U_{n+1} = U_n - h$, $U_0 = 1$. Thus

t	A: direct Euler	B: $1/U_n$
$1/4$	$5/4$	$4/3$
$1/2$	$105/64 = 1.640625$	2

Step 2: Compare with the exact solution. Separation gives $y = 1/(1-t)$, whose maximal interval containing zero is $(-\infty, 1)$. Algorithm B is exact at every grid node before 1, because its transformed equation has the constant slope -1 : Euler integrates that affine solution exactly. Algorithm A underestimates the two displayed exact values.

Step 3: Compare a single step. From the same positive state, the updates are

$$A(y) = y + hy^2, \quad B(y) = \frac{1}{1/y - h} = \frac{y}{1 - hy}.$$

Their exact difference is

$$B(y) - A(y) = \frac{h^2 y^3}{1 - hy} > 0 \quad (hy < 1).$$

For fixed y , the discrepancy is of order h^2 as $h \rightarrow 0$, although both are Euler approximations in their chosen coordinates. The nonlinear reciprocal map changes the finite-step rule.

Step 4: Respect the original blow-up. At $t = 1$, algorithm B has $U_4 = 0$, so its inverse is undefined, matching the exact pole. The direct recurrence uses only addition and multiplication and gives a finite real value after any fixed finite number of exact-arithmetic steps. Those values are numerical iterates, not proof of a classical solution through the pole.

The original solution tends to $+\infty$ as $t \uparrow 1$, so it cannot extend continuously across that time. Beyond 1, an inverse value from negative U_n would likewise belong to a disconnected formula branch, not a continuation of this IVP.

Question 7

For $y' = 1 - y$, $y(0) = 0$, use explicit Euler to reach $t = 1$ with N equal steps, $h = 1/N$. Assume exact arithmetic and define the signed error $E_n = Y_n - y(t_n)$. You may use $0 \leq e^{-h} - 1 + h \leq h^2/2$ for $0 < h \leq 1$.

Tasks

1. Find the exact solution and a closed formula for the Euler sequence. Determine the sign of E_n .
2. Derive an error recurrence using the one-step defect evaluated from the exact solution.
3. Prove $0 \leq E_n \leq (h/2)[1 - (1 - h)^n] \leq h/2$. Use the simpler last bound to choose an integer N guaranteeing endpoint error at most 0.005.
4. For your chosen N , calculate the actual endpoint error from the closed formulas. Explain the difference between a guaranteed bound and an observed error, and state which arithmetic assumption the proof uses.

Question 7 – Solution

Strategy. Exploit the contracting error recurrence to turn a local defect bound into a rigorous global guarantee.

Step 1: Solve both continuous and discrete problems. The exact solution is $y(t) = 1 - e^{-t}$. Euler gives $Y_{n+1} = (1 - h)Y_n + h$, so

$$\boxed{Y_n = 1 - (1 - h)^n}, \quad E_n = e^{-nh} - (1 - h)^n \geq 0.$$

For $0 < h < 1$, the sign follows from $1 - h < e^{-h}$; at $h = 1$, it follows directly for $n \geq 1$. At $n = 0$, both solutions are zero.

Step 2: Isolate the local defect. Starting Euler at the exact value at t_n , its overshoot of the exact next value is

$$d_n = y(t_n) + h[1 - y(t_n)] - y(t_n + h) = e^{-t_n}(e^{-h} - 1 + h).$$

Therefore $0 \leq d_n \leq h^2/2$, and subtracting the exact update from the numerical one gives

$$\boxed{E_{n+1} = (1 - h)E_n + d_n, \quad E_0 = 0}.$$

Step 3: Sum the propagated defects. Since $0 \leq 1 - h < 1$, iteration gives

$$0 \leq E_n \leq \frac{h^2}{2} \sum_{j=0}^{n-1} (1 - h)^j = \frac{h}{2} [1 - (1 - h)^n] \leq \frac{h}{2}.$$

Thus $1/(2N) \leq 0.005$ is sufficient. The smallest integer supplied by this simpler bound is $\boxed{N = 100}$, with $h = 0.01$. This is not a claim that no smaller N could meet the tolerance using a sharper analysis.

Step 4: Evaluate the actual error. For that grid,

$$\boxed{E_{100} = e^{-1} - (0.99)^{100} \approx 0.00184710 < 0.005}.$$

The proven bound controls the error without assuming a particular observed cancellation or leading asymptotic term. Its conservatism explains the smaller actual error. Both the recurrence and this guarantee assumed exact arithmetic; step-by-step rounding would add another error term and require a separate bound.

Question 8

For $y' = -y$, $y(0) = 1$, let Y_h denote the Euler approximation at $t = 1$ with equal step h . Compare $h = 1/2$ and $h = 1/4$. Assume exact arithmetic. For the extrapolation argument, suppose $Y_h = y(1) + Ch + O(h^2)$ with a coefficient C independent of h as $h \rightarrow 0$.

Tasks

1. Compute $Y_{1/2}$, $Y_{1/4}$, and $R = 2Y_{1/4} - Y_{1/2}$.
2. Use the stated error expansion to explain why R cancels the leading first-order error. Relate the difference between the two Euler values to an estimate of the finer-grid error.
3. Compare the actual errors of the two Euler values and R with the exact endpoint. Is it a contradiction if R lies outside the interval between the Euler values?
4. Test whether grid agreement certifies accuracy using a different IVP, $z' = \sin^2(4\pi t)$, $z(0) = 0$. Compute both Euler endpoints on the same grids and the exact value $z(1)$, and explain the failure of the grid-difference indicator.

Question 8 – Solution

Strategy. Separate asymptotic error cancellation from a rigorous certificate: two grids can share the same sampling blind spot.

Step 1: Compute the three approximations. For decay, $Y_h = (1 - h)^{1/h}$. Hence

$$Y_{1/2} = 1/4, \quad Y_{1/4} = 81/256, \quad R = 49/128 = 0.3828125.$$

Step 2: Cancel the leading error term. The assumed expansions give

$$2Y_{h/2} - Y_h = y(1) + O(h^2), \quad Y_{h/2} - Y_h = -Ch/2 + O(h^2).$$

Thus $|Y_{h/2} - Y_h|$ estimates the leading magnitude of the finer-grid error, while the extrapolated value removes that term. This argument is asymptotic; it supplies no explicit bound for the omitted terms on these particular grids.

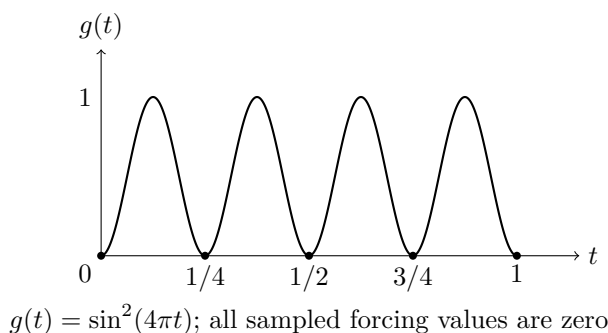
Step 3: Compare actual errors. The exact endpoint is e^{-1} . Absolute errors are

value	$Y_{1/2}$	$Y_{1/4}$	R
absolute error	0.117879	0.051473	0.014933

The finer-grid estimate is $17/256 = 0.06640625$, not its exact error. Both Euler values are below the truth, while R is above it. Extrapolation is not interpolation, so lying outside their interval is consistent with its construction and does not by itself invalidate it.

Step 4: Exhibit shared undersampling. At all left nodes of both specified grids, $\sin^2(4\pi t) = 0$. Both Euler approximations for $z(1)$ are therefore zero, and their difference is zero. But

$$z(1) = \int_0^1 \sin^2(4\pi t) dt = \frac{1}{2}.$$



The nonnegative forcing is missed between the sample nodes. Agreement of two grids does not certify accuracy; an explicit error bound or further analysis of the resolved time scales is needed.

Question 9

Consider $y' = -0.1y$, $y(0) = 1$, with Euler step $h = 1$. Let Z_n be the exact-arithmetic Euler sequence. A second implementation rounds each new value to the nearest tenth immediately after each step, calling the result Y_n . Use decimal rounding with ties away from zero; in particular, 0.45 rounds to 0.5. Assume the stated decimal rule is applied exactly, rather than relying on binary floating-point tie behavior.

Tasks

1. Derive the exact-arithmetic and rounded recurrences. Tabulate Y_n for $n = 0, 1, \dots, 8$ and compare its eventual behavior with Z_n and the exact solution.
2. Find every nonnegative multiple of 0.1 that is a fixed point of the rounded update.
3. Writing each rounding error as ρ_n , derive an error recurrence for $D_n = Y_n - Z_n$ and prove $|D_n| \leq 0.5(1 - 0.9^n)$.
4. Explain why a stable Euler amplification factor does not force a repeatedly rounded computation to converge to the true equilibrium. Distinguish rounding at every step from rounding only the reported final answer.

Question 9 – Solution

Strategy. Treat rounding as a new perturbation at each step; it can create fixed points absent from the exact-arithmetic recurrence.

Step 1: Compute both updates. Exact Euler gives $Z_{n+1} = 0.9Z_n$, so $Z_n = 0.9^n$. The rounded implementation uses $Y_{n+1} = \text{round}_{0.1}(0.9Y_n)$ and yields

n	0	1	2	3	4	5	6	7	8
Y_n	1.0	0.9	0.8	0.7	0.6	0.5	0.5	0.5	0.5

Once it reaches 0.5, its unrounded next value is 0.45, which rounds back to 0.5. Both Z_n and the exact solution $e^{-0.1t}$ tend to zero, whereas this implementation stays at 0.5.

Step 2: Identify all spurious fixed points. Write a nonnegative grid value as $Y = k/10$, with integer $k \geq 0$. In units of one tenth, the unrounded update is $0.9k$. For $k > 0$, rounding back to k occurs exactly when

$$k - \frac{1}{2} \leq 0.9k < k + \frac{1}{2},$$

which reduces to $k \leq 5$. Zero is also fixed. Thus the complete set is

$$\boxed{0, 0.1, 0.2, 0.3, 0.4, 0.5}.$$

Step 3: Bound accumulated rounding. Writing $Y_{n+1} = 0.9Y_n + \rho_n$ gives $|\rho_n| \leq 0.05$. Therefore

$$D_{n+1} = 0.9D_n + \rho_n, \quad D_0 = 0,$$

so

$$\boxed{|D_n| \leq 0.05 \sum_{j=0}^{n-1} 0.9^j = 0.5(1 - 0.9^n)}.$$

This bound approaches a nonzero limit, consistent with the observed plateau.

Step 4: Interpret the stable but biased computation. A factor 0.9 damps old perturbations, but fresh rounding occurs at every step and can balance the intended decrease. Stability does not remove continually injected errors. Rounding only the final reported answer would leave the preceding exact-arithmetic Euler trajectory unchanged and would introduce only one final reporting error, rather than altering subsequent slopes.

Question 10

Use explicit Euler to integrate $y' = -2y$ from $t = 1$ backward to $t = 0$. Begin with the benchmark terminal value $y(1) = e^{-2}$ and use time increments $\Delta t = -1/4$. This is the explicit Euler formula on a decreasing time grid, not the implicit method sometimes called backward Euler.

Tasks

1. Derive the update on the decreasing grid and compute the approximation at $t = 0$.
2. Find the benchmark exact solution, calculate the reconstruction error at 0, and sketch the numerical polygon with its direction of computation.
3. Replace the supplied terminal value by $e^{-2} + \delta$. Determine how this perturbation affects the numerical reconstruction and the exact reconstruction.
4. Now suppose e^{-2} is a reported terminal measurement with unknown error at most 0.001. Find the interval of possible exact initial values. Can reducing the Euler step alone guarantee reconstruction within 0.005 of the unknown true initial value? Explain.

Question 10 – Solution

Strategy. Track the sign of the time increment and distinguish discretization error from amplification of uncertain terminal data.

Step 1: March backward explicitly. Writing $t_j = 1 - j/4$, Euler gives

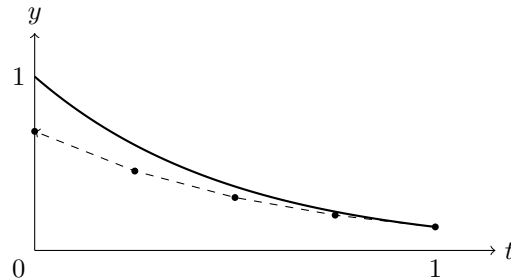
$$Y_{j+1} = Y_j - \frac{1}{4}(-2Y_j) = \frac{3}{2}Y_j.$$

After four steps,

$$Y(0) = \left(\frac{3}{2}\right)^4 e^{-2} = \frac{81}{16}e^{-2} \approx 0.685135.$$

Step 2: Compare with the exact benchmark. The exact solution is $y(t) = e^{-2t}$, giving $y(0) = 1$. Thus the numerical reconstruction error is

$$Y(0) - y(0) = \frac{81}{16}e^{-2} - 1 \approx -0.314865.$$



Solid: exact; dashed: Euler computed from $t = 1$ to $t = 0$

Each decreasing-time step amplifies the current value by 1.5. The exact amplification over a quarter-unit backward step is $e^{1/2} > 1.5$, explaining the numerical underestimate.

Step 3: Propagate a data perturbation. Linearity shows that replacing the terminal value by $e^{-2} + \delta$ changes the numerical reconstruction by

$$\left(\frac{81}{16}\right)\delta,$$

whereas exact backward evolution changes it by $e^2\delta$. With N backward steps of size magnitude $1/N$, the numerical amplification is $(1 + 2/N)^N$, which tends to e^2 , not to zero. Refinement improves the evolution calculation but does not eliminate sensitivity to data.

Step 4: Separate the uncertainty floor. The possible exact initial values form

$$[1 - 0.001e^2, 1 + 0.001e^2] \approx [0.992611, 1.007389].$$

Its half-width is about 0.007389, exceeding 0.005. No single reconstruction from the reported value can be within 0.005 of every possible true initial value in that interval. Reducing the step alone therefore cannot supply the requested guarantee; the terminal data would also need tighter uncertainty. Forward physical decay and backward data amplification describe different directions of evolution.