

Question 1

A tank initially holds 100 L of well-mixed solution containing 10 kg of salt. Solution containing 0.20 kg/L enters at 3 L/min, and the mixture leaves at 1 L/min. The tank has capacity 200 L. Assume instantaneous mixing, additive liquid volumes, and no salt precipitation or other losses. Let t be minutes after the flows begin.

Tasks

1. Derive initial-value models for the liquid volume $V(t)$ and salt mass $S(t)$, including units for all rate terms and the operating time before overflow.
2. Derive and solve an equation for the concentration $c(t) = S(t)/V(t)$. Explain why c' is not simply S'/V .
3. Find the first time the concentration reaches 0.15 kg/L and decide whether this happens before the tank reaches capacity.
4. Find the salt mass when the tank first becomes full. Sketch the concentration over the modeled operation and explain why continuing the same volume formula beyond that time changes the physical problem.

Question 1 – Solution

Strategy. Write separate liquid and salt balances, then use the quotient rule to model concentration in a changing volume.

Step 1: Balance volume and salt. The volume rate is $V' = 3 - 1 = 2$ L/min, so $V = 100 + 2t$ L. Capacity is reached at $t = 50$ min. Up to that time,

$$S' = 0.60 - \frac{S}{100 + 2t}, \quad S(0) = 10.$$

Both terms have units kg/min: salt enters at $(3)(0.20)$ and leaves at $(1)(S/V)$. The pre-overflow model applies for $0 \leq t < 50$, with a continuous value at 50.

Step 2: Derive the concentration equation. Since $S = cV$, $S' = Vc' + cV'$. Hence

$$(100 + 2t)c' = 0.60 - 3c, \quad c(0) = 0.10.$$

Using the integrating factor $(100 + 2t)^{3/2}$ and $c(0) = 0.10$ gives

$$c(t) = 0.20 - 0.10(100/(100 + 2t))^{3/2}.$$

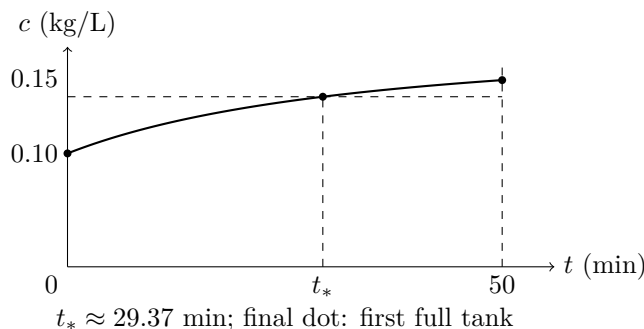
Indeed $c' = 3(0.20 - c)/V$, verifying the concentration balance. Omitting cV' would ignore the dilution caused by increasing volume.

Step 3: Locate the target and capacity values. The concentration increases strictly. Setting $c = 0.15$ gives

$$t_* = 50(2^{2/3} - 1) \approx 29.37 \text{ min} < 50 \text{ min}.$$

At capacity, $c(50) = 0.20 - 0.10/2^{3/2}$, so

$$S(50) = 200c(50) = 40 - 5\sqrt{2} \approx 32.93 \text{ kg}.$$



Step 4: Respect the operating boundary. The final point is the instant the tank becomes full. Beyond it, continued inflow would require overflow or a changed flow rate. Either changes the volume and salt balances, so the formula is not a model of the same operation after 50 min.

Question 2

An object is placed in a controlled chamber. Its temperature is initially 80°C , while the chamber temperature is $T_a(t) = 20 + 2t$ degrees Celsius, with t in minutes. Assume the object has a uniform temperature and obeys Newton's law of cooling with constant coefficient $k = 0.10 \text{ min}^{-1}$ throughout the experiment. The chamber warming is prescribed and unaffected by the object.

Tasks

1. Formulate the temperature IVP, including the sign of the heat-transfer term.
2. Solve it and verify the initial temperature and differential equation.
3. Find when the object stops cooling and determine its minimum temperature. Explain how the object can subsequently warm under the same law.
4. Sketch the object and chamber temperatures. Find their limiting difference and explain why their temperatures do not become equal permanently.

Question 2 – Solution

Strategy. Model the difference between the object and its time-dependent surroundings; the ambient temperature is not a constant equilibrium value.

Step 1: Form and solve the IVP. Newton's law gives

$$T' = -0.10(T - 20 - 2t), \quad T(0) = 80.$$

The right side is in degrees Celsius per minute. The object cools when it is hotter than the chamber and warms when it is colder. An integrating factor $e^{t/10}$ yields

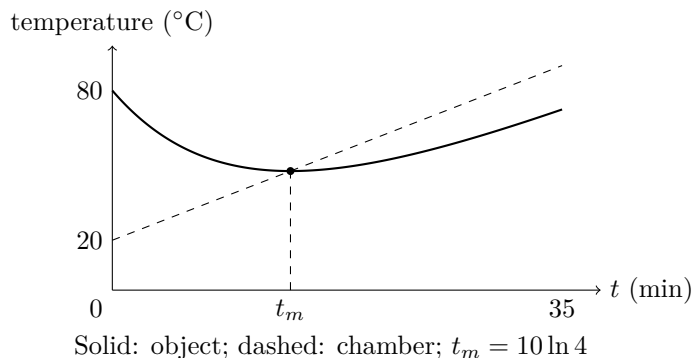
$$T(t) = 2t + 80e^{-t/10}, \quad t \geq 0.$$

Its derivative is $2 - 8e^{-t/10}$, which equals $-0.10(T - T_a)$, and its initial value is 80.

Step 2: Identify the turning time. The derivative vanishes when $e^{-t/10} = 1/4$, giving

$$t_m = 10 \ln 4 \approx 13.86 \text{ min}, \quad T_{\min} = 20 + 20 \ln 4 \approx 47.73^\circ\text{C}.$$

Since $T'' = 0.8e^{-t/10} > 0$, the derivative changes from negative to positive exactly once. At this time $T = T_a$; afterward the rising chamber is warmer than the object, so the heat-transfer term becomes positive.



Step 3: Interpret the long-time lag. The signed temperature difference is

$$T - T_a = 80e^{-t/10} - 20 \longrightarrow -20^\circ\text{C}.$$

The object eventually warms at almost $2^\circ\text{C}/\text{min}$. A temperature deficit of 20°C supplies that rate because $-0.10(-20) = 2$. Permanent equality would instead give zero heat-transfer rate, inconsistent with following an ambient temperature that keeps rising. The long-time conclusion is conditional on the stated linear ambient schedule and constant cooling coefficient remaining valid.

Question 3

A specimen contains a stable material of unknown mass B and an active material of mass $R(t)$ that disappears from the measured specimen at a rate proportional to R . Assume the stable material is unaffected and all disappearance products leave the measured mass. Thus the measured mass is $M(t) = B + R(t)$. Measurements, treated as exact, are

$$M(0) = 100 \text{ g}, \quad M(1) = 70 \text{ g}, \quad M(2) = 55 \text{ g},$$

where t is in days. Assume $B \geq 0$, $R(0) > 0$, and a positive constant disappearance coefficient.

Tasks

1. Formulate the active-mass IVP and the resulting measurement model.
2. Determine uniquely the stable mass, initial active mass, and disappearance coefficient from the data.
3. Find the active material's half-life and the time at which only 5 g of active material remain.
4. A technician fits exponential decay directly to the total mass using the first two measurements. Test that model against the third measurement and explain the modeling error.

Question 3 – Solution

Strategy. Separate the unchanging baseline from the decaying component. Differences of equally spaced measurements remove the baseline.

Step 1: Write the component and observation models. The active mass satisfies $R' = -kR$, where $k > 0$ has units day^{-1} . If $R(0) = R_0$, then

$$R = R_0 e^{-kt}, \quad M = B + R_0 e^{-kt}.$$

The stable material contributes to the measurement but not to the loss rate.

Step 2: Identify the parameters. Let $r = e^{-k}$, where one day is the time step, so $0 < r < 1$. The measured successive differences satisfy

$$-30 = R_0(r - 1), \quad -15 = R_0 r(r - 1).$$

Their ratio gives $r = 1/2$. Therefore

$$\boxed{R_0 = 60 \text{ g}, \quad B = 40 \text{ g}, \quad k = \ln 2 \text{ day}^{-1}}, \quad \boxed{M(t) = 40 + 60 2^{-t}}.$$

These values reproduce 100, 70, 55 g. They are unique under the assumptions because the nonzero first difference fixes r , then R_0 , then B . Also $R' = -kR$ verifies the rate law.

Step 3: Compute active-mass times. The active half-life is $\ln 2/k = \boxed{1 \text{ day}}$. The condition $R = 5$ gives $60 2^{-t} = 5$, hence

$$\boxed{t = \frac{\ln 12}{\ln 2} \approx 3.585 \text{ days}}.$$

The active mass is below 5 g after that time. The total mass then is 45 g, not 5 g.

Step 4: Test the incorrect observation model. A single exponential through the first two total-mass measurements is $\widetilde{M}(t) = 100(0.7)^t$. It predicts $\widetilde{M}(2) = 49$ g, contradicting the measured 55 g. It wrongly assigns the stable component the same disappearance law as the active material and predicts a zero limiting total mass instead of 40 g.

The model is intended for $t \geq 0$. Its inference depends on exact data, constant k , and a truly constant baseline; noisy data would require an estimation method rather than exact identification from three values.

Question 4

A 2 kg test package is released from rest. Take downward as positive and use $g = 10 \text{ m/s}^2$. Model drag as a force $-bv$ opposing its downward velocity v . Before a parachute opens at $t = 4 \text{ s}$, $b = 2 \text{ kg/s}$; afterward, $b = 10 \text{ kg/s}$. Assume the opening changes the drag coefficient instantly but delivers no impulse, and the package remains airborne through $t = 6 \text{ s}$.

Tasks

1. Derive the velocity equation in each phase from Newton's second law. State the initial and matching conditions.
2. Solve for $v(t)$ through 6 s and calculate the velocity immediately before and after opening.
3. Find the total downward distance traveled during the first 6 s.
4. Find the one-sided accelerations at opening and sketch the velocity. Explain why velocity is continuous although acceleration jumps, and state where a classical phase equation applies.

Question 4 – Solution

Strategy. Balance forces in each phase and carry velocity continuously across a coefficient change; integrate velocity to obtain displacement.

Step 1: Model and match the phases. Newton's law gives $2v' = 20 - bv$, with forces in newtons. Thus $v' = 10 - v$ before opening and $v' = 10 - 5v$ afterward. The initial value is $v(0) = 0$, and no impulse means $v(4^-) = v(4^+) = v_4$.

Step 2: Solve both initial-value problems. The solutions are

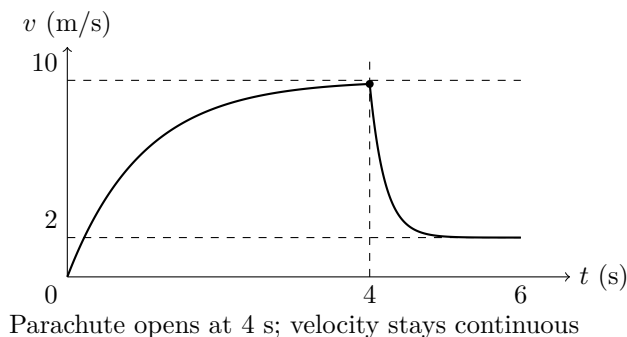
$$v(t) = \begin{cases} 10(1 - e^{-t}), & 0 \leq t \leq 4, \\ 2 + (v_4 - 2)e^{-5(t-4)}, & 4 \leq t \leq 6, \end{cases} \quad v_4 = 10(1 - e^{-4}) \approx 9.817 \text{ m/s}.$$

Each expression satisfies its phase equation, and both give v_4 at the switch. The phase terminal speeds are 10 and 2 m/s, respectively.

Step 3: Integrate the distance. Velocity stays positive, so downward displacement equals distance traveled:

$$d = 10(3 + e^{-4}) + 4 + \frac{8 - 10e^{-4}}{5}(1 - e^{-10}) \approx 35.75 \text{ m}.$$

The first term integrates the pre-opening velocity over $[0, 4]$; the remaining terms integrate the post-opening velocity over $[4, 6]$.



Step 4: Examine the acceleration jump. The one-sided accelerations are

$$v'(4^-) = 10e^{-4} \approx 0.183 \text{ m/s}^2, \quad v'(4^+) = -40 + 50e^{-4} \approx -39.08 \text{ m/s}^2.$$

Changing a finite force changes acceleration without an instantaneous velocity jump. The velocity is continuous and piecewise differentiable, but is not differentiable at the idealized switch. Each classical phase equation applies away from $t = 4$; the switch is handled by continuity, not by requiring a single derivative there.

Question 5

A series resistor-capacitor circuit has resistance $R = 1000 \, \Omega$ and capacitance $C = 0.001 \, \text{F}$. Its capacitor voltage is $U(t)$, charge is $q = CU$, and current is $I = q'$. The source voltage $E(t)$ obeys the loop relation $E = RI + U$. Initially the capacitor is uncharged. Time is in seconds.

A designer wants the exact voltage trajectory

$$U_d(t) = 6(1 - e^{-t/2}) \, \text{V}, \quad t \geq 0,$$

but the available source must satisfy $0 \leq E(t) \leq 5 \, \text{V}$. Ideal circuit laws are assumed; capacitor energy is $W = \frac{1}{2}CU^2$.

Tasks

1. Derive a first-order IVP for the capacitor voltage and identify the time constant, with units.
2. Find the unique source voltage and the current required to produce the desired trajectory exactly.
3. Determine the longest interval starting at 0 on which this trajectory respects the source bound. Find the capacitor voltage at the limiting time.
4. Calculate the stored energy at that time. Explain why simply fixing the source at 5 V afterward cannot continue the same prescribed trajectory.

Question 5 – Solution

Strategy. Use the circuit balance backward: a specified capacitor trajectory determines its current and therefore the required source.

Step 1: Derive the voltage model. Since $I = CU'$, the loop equation gives

$$RCU' + U = E(t), \quad U(0) = 0.$$

Here $RC = (1000)(0.001) = 1$ s is the time constant. With time numerically measured in seconds, the model is $U' + U = E$; the derivative coefficient carries the one-second time constant.

Step 2: Find the required input and current. The desired derivative is $U'_d = 3e^{-t/2}$ V/s, so

$$E(t) = 6 - 3e^{-t/2} \text{ V}, \quad I(t) = 0.003e^{-t/2} \text{ A}.$$

Substitution gives $RCU'_d + U_d = E$ and $U_d(0) = 0$. The prescribed U_d fixes its derivative, so the loop relation leaves no other possible source trajectory. In particular, the initial current is 3 mA and the required initial source is 3 V.

Step 3: Apply the source constraint. The required source increases strictly from 3 V toward 6 V. It first reaches 5 V when $e^{-t/2} = 1/3$. Therefore the full feasible initial interval is

$$0 \leq t \leq t_* = 2 \ln 3 \approx 2.197 \text{ s}, \quad U_d(t_*) = 4 \text{ V}.$$

Equality at the limiting time is allowed by the given source bound. Beyond it the unique required source exceeds 5 V.

Step 4: Compute energy and interpret saturation. At the limiting time,

$$W = \frac{1}{2}(0.001)(4)^2 = 0.008 \text{ J}.$$

A source fixed at 5 V afterward would instead produce $U = 5 - e^{-(t-t_*)}$ V, starting from 4 V. It tends to 5 V, whereas the desired trajectory tends to 6 V. Thus the circuit can continue operating, but it cannot continue this exact target under the source constraint. The feasible interval is a hardware limitation, not a singularity of the ideal differential equation.

Question 6

An inverted conical tank has height 4 m and top radius 2 m. It initially contains water to the brim. Let $h(t)$ be water depth above the outlet, with t in seconds. Assume horizontal water surfaces and similar conical cross-sections. The outflow rate is prescribed by

$$Q(h) = \kappa\sqrt{h}, \quad \kappa = 0.05 \text{ m}^{5/2}/\text{s},$$

while $h > 0$. There is no inflow. Ignore changes in the discharge coefficient and stop outflow once the tank is empty.

Tasks

1. Derive the volume as a function of depth and use volume conservation to form an IVP for h .
2. Solve for the depth until emptying and find the emptying time.
3. Find the fractions of the emptying time needed to reach half the original depth and half the original volume. Explain why these are different.
4. Sketch the physical depth, including the empty state. Decide whether depth and volume have continuously differentiable extensions through emptying, and explain the scope of the divided depth equation.

Question 6 – Solution

Strategy. The water surface area changes with depth. Convert a volume-loss law into a depth equation before separating variables.

Step 1: Use the conical geometry. Similarity gives $r = h/2$, so $V = \pi h^3/12$ and

$$\frac{\pi h^2}{4} h' = -0.05\sqrt{h}, \quad h' = -\frac{0.20}{\pi} h^{-3/2}, \quad h(0) = 4.$$

The first equality has units m^3/s . Division by h^2 is valid only while water remains.

Step 2: Integrate to emptying. Integrating $h^{3/2} dh = -(0.20/\pi) dt$ gives $h^{5/2} = 32 - t/(2\pi)$, hence

$$h(t) = (32 - t/(2\pi))^{2/5}, \quad 0 \leq t < t_e, \quad t_e = 64\pi \approx 201.06 \text{ s}.$$

Equivalently, $h = 4(1 - t/t_e)^{2/5}$. Differentiation verifies the depth equation; the physical depth is set to zero for $t \geq t_e$.

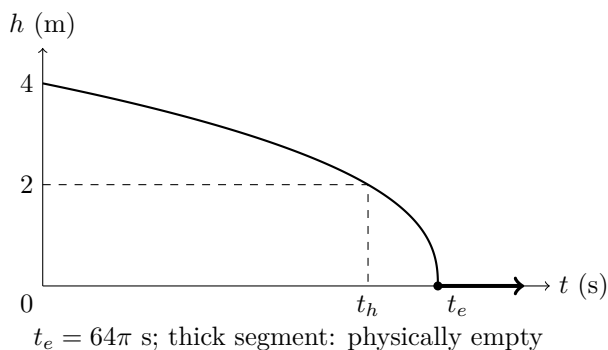
Step 3: Compare depth and volume targets. At $h = 2$ m, $h/4 = 1/2$, giving

$$t_h/t_e = 1 - 2^{-5/2} \approx 0.8232.$$

At half volume, $(h/4)^3 = 1/2$, so

$$t_V/t_e = 1 - 2^{-5/6} \approx 0.4388.$$

Half depth leaves only one eighth of the original volume, so it takes substantially longer than losing half the volume.



Step 4: Interpret the endpoint. As $t \uparrow t_e$, $h' \rightarrow -\infty$, so the zero continuation of depth is continuous but not differentiable at emptying. However, $V' = -0.05\sqrt{h} \rightarrow 0$, agreeing with $V' = 0$ afterward; volume has a continuously differentiable zero continuation. The divided depth equation is a wet-tank model and is undefined at $h = 0$. It must not be continued algebraically into a fictitious refilling branch.

Question 7

A well-mixed pond maintains volume 1000 m^3 . Water enters and leaves at $100 \text{ m}^3/\text{day}$. The entering water has a constant pollutant concentration $u \geq 0 \text{ kg/m}^3$, which is to be selected. Pollutant also disappears inside the pond at a rate equal to 0.10 times its current mass per day. Initially the pond concentration is 0.08 kg/m^3 .

For this hypothetical design problem, require the pond concentration to stay at or below 0.10 kg/m^3 throughout the first five days. Assume perfect mixing and that disappearance does not change the liquid volume.

Tasks

1. Derive the pollutant mass balance and the concentration IVP, keeping flushing and internal disappearance distinct.
2. Solve for the concentration in terms of u and verify the initial condition.
3. Determine the largest constant incoming concentration satisfying the five-day requirement, and justify checking the entire time interval.
4. Find the largest constant incoming concentration that would satisfy the same bound for all future time. Explain why it differs from the finite-horizon answer.

Question 7 – Solution

Strategy. Separate input, outflow and internal loss, then use the solution's monotonicity to turn a whole-interval constraint into a justified endpoint test.

Step 1: Write the balances. If M is pollutant mass in kg and $c = M/1000$, then

$$M' = 100u - 100\frac{M}{1000} - 0.10M, \quad M(0) = 80.$$

Each term is in kg/day. Dividing by the constant volume gives

$$c' = 0.10u - 0.20c, \quad c(0) = 0.08.$$

The total loss coefficient combines flushing at 0.10 day^{-1} with internal disappearance at the same rate.

Step 2: Solve and identify monotonicity. An integrating factor $e^{0.20t}$ gives

$$c(t) = \frac{u}{2} + (0.08 - u/2)e^{-0.20t}, \quad t \geq 0.$$

Its initial value is 0.08, and differentiating verifies $c' = 0.10u - 0.20c$. It is decreasing when $u < 0.16$, constant when $u = 0.16$, and increasing when $u > 0.16$.

Step 3: Enforce the five-day bound. For $u \leq 0.16$, the maximum is the acceptable initial concentration. For $u > 0.16$, the maximum on $[0, 5]$ is $c(5)$. Therefore the largest allowed input is

$$u_5 = \frac{2(0.10 - 0.08e^{-1})}{1 - e^{-1}} \approx 0.2233 \text{ kg/m}^3.$$

This value exceeds 0.16, so its concentration rises monotonically from 0.08 to exactly 0.10. Any larger u fails at day five, proving optimality rather than just feasibility.

Step 4: Compare a permanent operating requirement. The limiting pond concentration is $u/2$. A bound valid for all $t \geq 0$ requires and is guaranteed by

$$u \leq 0.20 \text{ kg/m}^3.$$

For these inputs, $c(t)$ lies between its initial value and $u/2$. If $u > 0.20$, the limit exceeds the bound, so it must eventually be violated. The larger finite-horizon input uses the initially cleaner pond as temporary dilution capacity; it cannot be continued indefinitely under the same requirement.

Question 8

An isolated laboratory population is modeled by a per-capita growth rate that decreases linearly with population size:

$$P' = P(a - bP), \quad a > 0, b > 0.$$

Treat P as a continuous number of individuals, with time in days. Under unchanged laboratory conditions, the instantaneous total growth rate is 20 individuals/day both when $P = 100$ and when $P = 200$. A new culture begins with $P(0) = 100$.

Tasks

1. Identify a and b , including units, from the two instantaneous rate observations.
2. Solve the new culture's IVP and verify the result. State the assumptions supporting use of the fitted parameters in this culture.
3. Find when the population first reaches 250 individuals and explain why this is a unique future time.
4. An exponential model is fitted only to the initial population and its initial growth rate. Compare its rate prediction at $P = 200$ with the second observation, and explain what that discrepancy says about the model.

Question 8 – Solution

Strategy. Convert total growth observations to per-capita rates before estimating the crowding coefficient.

Step 1: Calibrate the rate law. The observations give

$$a - 100b = 0.20, \quad a - 200b = 0.10.$$

Subtracting yields

$$a = 0.30 \text{ day}^{-1}, \quad b = 0.001 \text{ individual}^{-1}\text{day}^{-1}.$$

The model becomes $P' = 0.30P(1 - P/300)$. Both specified populations give total rate 20 individuals/day.

Step 2: Solve the IVP. Separation for the branch $0 < P < 300$ gives

$$\ln \frac{P}{300 - P} = 0.30t + C.$$

Using $P(0) = 100$ gives $C = -\ln 2$ and

$$P(t) = \frac{300}{1 + 2e^{-0.30t}}, \quad t \geq 0.$$

If $z = 2e^{-0.30t}$, then $P' = 90z/(1 + z)^2 = 0.30P(1 - P/300)$, and the initial value is 100. Applying the calibration assumes the same fixed resources and crowding law, no migration, and a population scale for which the continuous approximation is appropriate.

Step 3: Locate the target. Setting $P = 250$ yields $e^{-0.30t} = 1/10$, so

$$t = \frac{\ln 10}{0.30} \approx 7.675 \text{ days}.$$

The formula increases strictly from 100 toward 300 for $t \geq 0$, so 250 is attained exactly once. Its denominator remains positive throughout this physical time range.

Step 4: Test the competing exponential model. For $P' = rP$, the initial observation forces $r = 20/100 = 0.20 \text{ day}^{-1}$. At $P = 200$, that model predicts 40 individuals/day, double the observed rate. Matching one population and one instantaneous slope does not validate a constant per-capita growth law.

The second observation distinguishes the two proposed laws. It supports the calibrated crowding model over this exponential alternative under the stated exact-data assumptions, but two rate observations alone do not establish that the model remains accurate at every population size.

Question 9

A spherical droplet has constant density ρ and radius $r(t)$. Assume its mass disappears at a rate proportional to its current surface area, with a constant coefficient $k > 0$:

$$M = \frac{4}{3}\pi\rho r^3, \quad M' = -k(4\pi r^2)$$

while the droplet exists. Radius decreases from 3 mm to 2 mm during the first 5 minutes. Ignore changes in shape, density and environmental conditions.

Tasks

1. Derive the radius IVP from the mass balance and identify k/ρ and its units from the observations. Can k itself be found without knowing ρ ?
2. Find the extinction time and the fraction of original mass remaining at any time before extinction.
3. Determine when half the original mass remains; compare this with the time at which the radius is halved.
4. Fit an exponential mass-loss model with the same initial mass and initial mass-loss rate. Compare the two predictions at 10 minutes and discuss their extinction predictions.

Question 9 – Solution

Strategy. Differentiate the geometric mass formula before applying the surface-loss law. A constant radius-loss rate need not imply exponential mass loss.

Step 1: Reduce the balance to radius. For $r > 0$, the chain rule gives

$$4\pi\rho r^2 r' = -4\pi k r^2, \quad r' = -k/\rho, \quad r(0) = 3 \text{ mm}.$$

The observed radius loss is 1 mm in 5 min, hence

$$\boxed{k/\rho = 0.20 \text{ mm/min}}, \quad \boxed{r(t) = 3 - 0.20t \text{ mm}}.$$

If density is in mass/mm^3 , k has units $\text{mass}/(\text{mm}^2 \text{ min})$. The observations identify only the ratio; a density value is needed to determine k itself.

Step 2: Determine mass and extinction. The radius reaches zero at $\boxed{t_e = 15 \text{ min}}$. Since mass is proportional to r^3 ,

$$\boxed{\frac{M(t)}{M(0)} = (1 - t/15)^3}, \quad 0 \leq t \leq 15.$$

The physical mass and radius remain zero afterward. Extending the linear radius formula past 15 would produce negative radius and is not part of the droplet model. Differentiating the cubic mass formula recovers the original surface-loss balance while $r > 0$.

Step 3: Compare fractional targets. Half mass occurs when $(1 - t/15)^3 = 1/2$, giving

$$\boxed{t_{1/2,M} = 15(1 - 2^{-1/3}) \approx 3.094 \text{ min}}.$$

Half radius occurs at $t = 7.5 \text{ min}$, when only $(1/2)^3 = 1/8$ of the initial mass remains. Geometric scaling makes these distinct events.

Step 4: Test an exponential alternative. The initial relative mass slope from the cubic formula is $M'(0)/M(0) = -1/5 \text{ min}^{-1}$. The matching exponential is therefore

$$M_{\text{exp}}(t) = M(0)e^{-t/5}.$$

At 10 min the surface-loss model predicts $M/M(0) = 1/27 \approx 0.0370$, whereas the exponential predicts $e^{-2} \approx 0.1353$. The exponential never reaches zero at finite time; the geometric model empties at 15 min. Agreement of initial value and slope does not imply agreement of the physical loss mechanism or its later predictions.

Question 10

A reservoir initially contains 10 L of usable liquid. During a four-minute operation, a pump supplies a constant rate $u \geq 0$ L/min, scheduled demand removes liquid at rate $d(t) = 2 + t$ L/min, and leakage removes liquid at a rate equal to 0.50 times the current volume per minute. Here t is in minutes; the slope in $d(t)$ is 1 L/min².

The reservoir must retain at least 4 L at every time $0 \leq t \leq 4$. Assume no capacity restriction, no other flows, and that the stated demand and leakage laws apply while liquid remains.

Tasks

1. Derive and solve the volume IVP in terms of u .
2. Find the smallest constant pump rate that satisfies the reserve requirement throughout the operation. Justify feasibility over the whole interval, not only at its endpoint.
3. For this rate, calculate the cumulative leaked volume by $t = 4$ using an overall volume balance.
4. Compare the answer with the minimum pump rate obtained by ignoring leakage. Sketch the optimal volume and reserve level, and explain why the no-leakage design cannot meet the original requirement.

Question 10 – Solution

Strategy. Solve the storage balance, obtain a necessary pump rate from the final reserve, and then verify that this candidate satisfies the full time constraint.

Step 1: Model and solve the storage balance. Input minus demand minus leakage gives

$$S' = u - (2 + t) - 0.50S, \quad S(0) = 10,$$

with all rates in L/min. A particular solution is $2u - 2t$; adding $Ae^{-t/2}$ and applying $S(0) = 10$ gives

$$S(t) = 2u - 2t + (10 - 2u)e^{-t/2}.$$

It has initial value 10, and differentiation gives the required balance on any time interval on which the physical operation remains valid.

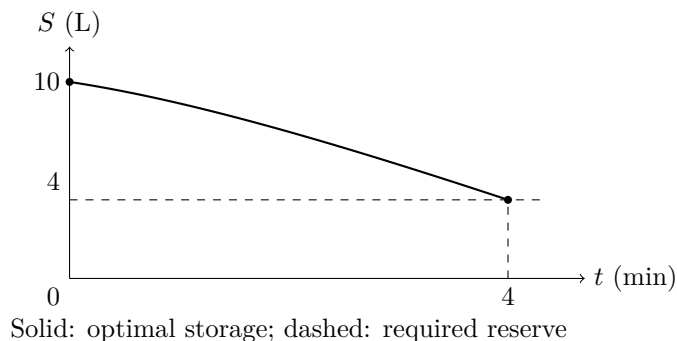
Step 2: Find and verify the minimum rate. Since $S(4) \geq 4$ is necessary,

$$u \geq u_* = \frac{12 - 10e^{-2}}{2(1 - e^{-2})} = 6 + \frac{1}{e^2 - 1} \approx 6.157 \text{ L/min}.$$

For this candidate, $6 < u_* < 7$ and

$$S'(t) = -2 + (u_* - 5)e^{-t/2} < 0 \quad (0 \leq t \leq 4).$$

Thus S decreases from 10 to exactly 4 L and never violates the reserve. The candidate is feasible, and every smaller rate fails the necessary final condition, proving optimality.



Step 3: Recover cumulative leakage. The scheduled demand totals $\int_0^4 (2 + t) dt = 16$ L. Initial storage plus pumped volume equals final storage plus demand plus leakage, so

$$L_{\text{leak}} = 10 + 4u_* - 16 - 4 = 4u_* - 10 \approx 14.63 \text{ L}.$$

Step 4: Compare the incomplete model. Ignoring leakage gives $S_0 = 10 + (u - 2)t - t^2/2$. This concave function has its minimum on $[0, 4]$ at an endpoint; $S_0(4) \geq 4$ gives the no-leakage minimum $u = 2.5$ L/min. That rate is below u_* and fails the actual reserve constraint. It may eventually exhaust the actual reservoir, so negative values of its unconstrained storage formula would not describe physical stored liquid. The optimal design above stays positive and avoids that limitation.