

Question 1

Consider the initial-value problem

$$y' = y^2, \quad y(0) = 2.$$

A student argues: “The right-hand side and its derivative with respect to y are continuous everywhere, so the solution must exist for every real x .”

Tasks

1. State what local existence and uniqueness actually guarantee for this IVP, and identify the gap in the student’s argument.
2. Solve the IVP and verify the initial value and differential equation.
3. Find its maximal open interval of validity containing 0, and prove that a classical solution cannot extend it past its finite endpoint.
4. The same rational expression is defined on another open interval. Explain why that branch is not a continuation of this IVP solution.

Here a classical solution is a real, continuously differentiable function satisfying the equation at every point of its interval. A maximal interval cannot be enlarged while extending the same solution of the given equation.

Question 1 – Solution

Strategy. Smoothness of the differential equation gives a local theorem; the explicit solution reveals whether its values remain finite.

Step 1: Separate local and global conclusions. Here $f(x, y) = y^2$ and $f_y(x, y) = 2y$ are continuous on the whole plane. The local theorem guarantees a solution on some open interval around 0 and uniqueness of solutions through $(0, 2)$ on their common interval. It does not guarantee a solution on the entire x -axis: growth in y can produce a finite-time blow-up.

Step 2: Solve and verify. Near the initial point $y \neq 0$, so separation gives

$$-\frac{1}{y} = x + C, \quad C = -\frac{1}{2}, \quad \boxed{y(x) = \frac{2}{1 - 2x}}.$$

Directly, $y(0) = 2$ and $y' = 4/(1 - 2x)^2 = y^2$. The expression therefore defines the selected solution at every $x < 1/2$.

Step 3: Establish maximality. The component of the formula's domain containing 0 is

$$\boxed{I = (-\infty, 1/2)}.$$

As $x \uparrow 1/2$, $y(x) \rightarrow +\infty$. Any extension to a larger interval would include $1/2$ as an interior point and would have a finite, continuous value there. That contradicts this limit. There is no finite obstruction to the left.

Step 4: Distinguish the disconnected branch. The expression also solves the differential equation on $(1/2, \infty)$, where it is negative. That interval does not contain the initial point. The union of the two components is not an interval, and no classical solution can connect them through the pole. An algebraic formula on both sides of a singularity is not a continuation across it.

Question 2

For a real parameter a , consider the equation in the form written:

$$y' + \frac{2x}{x^2 - 9}y = \frac{1}{x^2 - 9}, \quad y(0) = a.$$

The original equation is defined only for $x \neq -3, 3$.

Tasks

1. Use the linear existence theorem to identify an interval on which a unique solution is guaranteed for every a .
2. Find and verify the solution formula in terms of a .
3. Determine which values of a make a singularity of that formula removable, and compute the corresponding finite limit.
4. Determine the maximal interval for the original IVP for every a . Explain whether multiplying the equation by $x^2 - 9$ changes what it means to extend a solution through $x = \pm 3$.

Question 2 – Solution

Strategy. Keep the domain of the given equation separate from the domain of a simplified formula or a multiplied equation.

Step 1: Locate the coefficient interval. Both coefficients are continuous on $(-3, 3)$, the largest coefficient interval containing 0. The linear theorem guarantees a unique solution throughout that interval for every real a .

Step 2: Integrate and verify. On this interval, multiplication and the product rule yield

$$((x^2 - 9)y)' = 1, \quad (x^2 - 9)y = x - 9a.$$

Thus

$$y(x) = \frac{x - 9a}{x^2 - 9}.$$

The value at 0 is a , and differentiating $(x^2 - 9)y = x - 9a$ verifies the differential equation whenever $x \neq \pm 3$.

Step 3: Examine cancellation. At $x = 3$, the numerator vanishes exactly when $a = 1/3$. Then

$$y = \frac{1}{x + 3} \quad (x \neq \pm 3), \quad \lim_{x \rightarrow 3} y = \frac{1}{6}.$$

At $x = -3$, cancellation occurs exactly when $a = -1/3$. Then

$$y = \frac{1}{x - 3} \quad (x \neq \pm 3), \quad \lim_{x \rightarrow -3} y = -\frac{1}{6}.$$

No value of a cancels both factors. For other parameter values the corresponding endpoint is a pole.

Step 4: Use the original domain. For every a , the answer remains $I = (-3, 3)$. Even a finite removable limit does not define the original coefficients at an excluded point.

The multiplied equation $(x^2 - 9)y' + 2xy = 1$ is defined at $x = \pm 3$ and imposes algebraic conditions there. It is equivalent to the original equation only away from those points. A canceled formula can extend through one endpoint as a solution of the multiplied equation, but that is an extension for a different domain of definition.

Question 3

Consider

$$y' = \frac{e^{-y}}{x-1}, \quad y(2) = \ln 2,$$

with $x \neq 1$ and real y .

Tasks

1. Use $u = e^y$ to solve the IVP, explicitly enforcing the range of this substitution.
2. Determine the maximal interval containing 2; compare its left endpoint with the singular point of the coefficient $1/(x-1)$.
3. Verify the solution and calculate its limiting behavior at both ends of its maximal interval.
4. Explain why neither continuing the linear transformed solution through a zero nor replacing $\ln u$ by $\ln |u|$ produces a continuation of the original solution.

Question 3 – Solution

Strategy. The exponential substitution is invertible only for $u > 0$. Its range can restrict the interval before a coefficient singularity is reached.

Step 1: Transform and invert. On the component $x > 1$ containing 2,

$$u' = e^y y' = \frac{1}{x-1}, \quad u = \ln(x-1) + C.$$

Since $u(2) = 2$, $C = 2$. Therefore

$$y = \ln(2 + \ln(x-1))$$

is a real solution only where $2 + \ln(x-1) > 0$.

Step 2: Find the interval. The positivity condition is $x > 1 + e^{-2}$. Let $\alpha = 1 + e^{-2}$; then

$$I = (\alpha, \infty).$$

In particular, $1 < \alpha < 2$. The solution fails before reaching $x = 1$, even though the original right-hand side is smooth at every finite point (α, y) .

Step 3: Verify and analyze the ends. Writing $u = 2 + \ln(x-1) > 0$ gives

$$y' = \frac{1}{(x-1)u} = \frac{e^{-y}}{x-1}, \quad y(2) = \ln 2.$$

As $x \downarrow \alpha$, $u \downarrow 0$ and $y \rightarrow -\infty$, preventing a finite continuous extension. As $x \rightarrow \infty$, $u \rightarrow \infty$ and $y \rightarrow \infty$, but this occurs only at an infinite endpoint and does not shorten the interval.

Step 4: Reject an invalid inverse. The transformed function continues on $(1, \infty)$ but becomes negative on $(1, \alpha)$. No real y can have $e^y = u \leq 0$.

On that negative portion, $z = \ln |u|$ would satisfy $z' = 1/((x-1)u) < 0$, whereas $e^{-z}/(x-1) = 1/((x-1)|u|) > 0$. Thus the absolute-value replacement fails the original equation and cannot repair the lost exponential range.

Question 4

Consider the real IVP

$$y' = -\frac{x}{y}, \quad y(0) = 2,$$

whose equation is defined only when $y \neq 0$.

Tasks

1. Obtain an implicit relation and select the explicit branch determined by the initial condition.
2. Find the maximal interval and the limits of both y and y' at its finite endpoints.
3. Decide whether either endpoint can be included, or whether switching to the other square-root branch gives a continuation as a classical solution $y(x)$.
4. Sketch the selected solution and mark its excluded endpoints. Explain why bounded values of y do not guarantee continuation in this example.

Question 4 – Solution

Strategy. Track the forbidden value $y = 0$ and the slope, rather than checking only whether the solution stays bounded.

Step 1: Select the branch. Multiplying by y and integrating gives

$$yy' = -x, \quad x^2 + y^2 = 4.$$

Continuity and $y(0) = 2$ select

$$y = \sqrt{4 - x^2}.$$

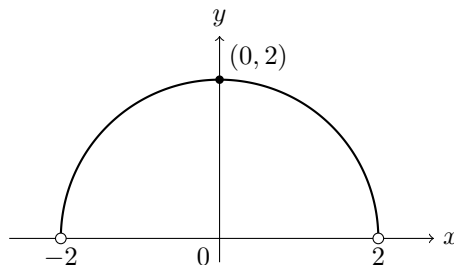
Its positive branch has $y' = -x/\sqrt{4 - x^2}$ and directly satisfies the IVP for $-2 < x < 2$.

Step 2: Test both endpoints. Thus $I = (-2, 2)$, and

	$x \downarrow -2$	$x \uparrow 2$
$y(x)$	0^+	0^+
$y'(x)$	$+\infty$	$-\infty$

At either endpoint the original right-hand side is undefined, and a continuously differentiable extension would also require a finite derivative.

Step 3: Exclude branch switching. The negative square root fails the initial condition. Joining the two semicircles at an endpoint still passes through $y = 0$, and the circle has no real points with $|x| > 2$. Following the circle as a parametrized curve is not an extension as a differentiable graph $y(x)$ on a larger interval.



Step 4: Identify the obstruction. Although $0 < y \leq 2$, the graph approaches the boundary of the equation's domain. Boundedness alone does not keep it inside a region where the differential equation is defined and regular. The open circles indicate excluded endpoints, not additional solution values.

Question 5

Consider

$$y' = \frac{1}{3y^2 - 1}, \quad y(0) = 0,$$

with $y \neq \pm 1/\sqrt{3}$.

Tasks

1. Derive an implicit equation and identify the portion that defines the IVP's branch as a function of x .
2. Determine the exact maximal interval containing 0, using monotonicity rather than a cubic-root formula.
3. Find the one-sided limits of y and y' at both endpoints and justify maximality.
4. Sketch the selected branch, and explain why other roots of the same cubic cannot be spliced onto it to extend this classical solution.

Question 5 – Solution

Strategy. Regard x as a function of y . Its turning points identify where the selected inverse branch stops being a regular graph.

Step 1: Integrate without solving the cubic. The chain rule gives

$$\frac{d}{dx}(y^3 - y) = (3y^2 - 1)y' = 1, \quad \boxed{x = y^3 - y}.$$

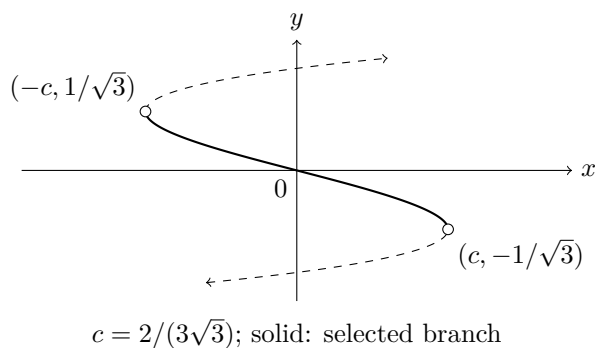
The initial value sets the integration constant to zero. Put $h(y) = y^3 - y$. On $-1/\sqrt{3} < y < 1/\sqrt{3}$, $h'(y) < 0$, so h is strictly decreasing and has a differentiable inverse there.

Step 2: Map the branch endpoints. Let $c = 2/(3\sqrt{3})$. Then $h(1/\sqrt{3}) = -c$ and $h(-1/\sqrt{3}) = c$. Hence the inverse through $(0, 0)$ exists exactly on

$$\boxed{I = (-c, c) = \left(-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)}.$$

Differentiating its defining relation recovers $y' = 1/(3y^2 - 1)$ and verifies the original equation on this branch.

Step 3: Analyze the folds. As $x \downarrow -c$, $y \rightarrow 1/\sqrt{3}$; as $x \uparrow c$, $y \rightarrow -1/\sqrt{3}$. In both cases $3y^2 - 1 \rightarrow 0^-$, so $y' \rightarrow -\infty$. Both finite limiting values are excluded by the original equation.



Step 4: Reject a change of root. The dashed pieces show other parts of the algebraic curve. A continuous switch at a fold must include an excluded value of y with unbounded slope; a switch to a different root at that x would be discontinuous. Neither gives a classical extension. The cubic relation alone does not specify the IVP's interval or branch.

Question 6

Consider the IVP

$$y' = 3|y|^{2/3}, \quad y(0) = 0.$$

For finite real numbers $a \leq 0 \leq b$, define

$$Y_{a,b}(x) = \begin{cases} (x-a)^3, & x < a, \\ 0, & a \leq x \leq b, \\ (x-b)^3, & x > b. \end{cases}$$

Tasks

1. Verify the equation and continuous differentiability of $Y_{a,b}$, including both joining points.
2. Give two distinct global solutions satisfying the initial condition and determine their maximal intervals.
3. Explain why a restriction of $Y_{-1,1}$ to $(-1/2, 1/2)$ is not maximal, even though it solves the IVP there.
4. Sketch $Y_{-1,1}$ and identify precisely which local uniqueness hypothesis fails. Does a maximal interval imply a unique solution?

Question 6 – Solution

Strategy. Check continuation at the joins directly. The length of a solution's interval and uniqueness of the solution are different questions.

Step 1: Verify all three pieces and the joins. For $x < a$,

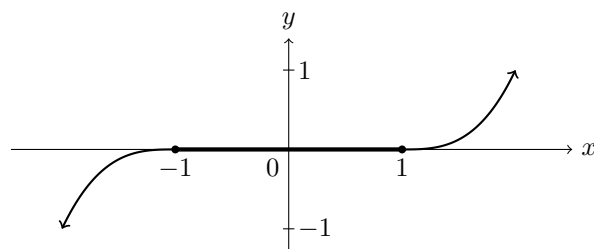
$$Y'_{a,b} = 3(x-a)^2 = 3|(x-a)^3|^{2/3},$$

and the same computation holds with b for $x > b$. On the middle piece both sides vanish.

At $x = a$, the values from both sides are 0, and the one-sided derivatives tend to 0. The derivative at the join itself is 0 by the difference quotient. The same holds at b , including the case $a = b = 0$. Thus $Y_{a,b}$ is continuously differentiable on $\mathbb{R}$, satisfies the equation everywhere, and has $Y_{a,b}(0) = 0$.

Step 2: Exhibit distinct maximal solutions. Two examples are $y(x) = 0$ for all x and $y(x) = x^3 = Y_{0,0}(x)$. Their maximal intervals are both $\boxed{\mathbb{R}}$: no larger real interval exists. The function $Y_{-1,1}$ is a third global solution.

Step 3: Separate a restriction from a maximal solution. On $(-1/2, 1/2)$, $Y_{-1,1} = 0$, but it extends as $Y_{-1,1}$ to the whole real line. Its restriction is therefore not maximal. That restriction can also be extended as the identically zero solution; an extension need not be unique here.



Solid graph: $Y_{-1,1}$; thick segment: $y = 0$

Step 4: Locate the failed uniqueness condition. The right-hand side is continuous, but it is not locally Lipschitz in y at 0:

$$\frac{|3|y|^{2/3} - 0|}{|y - 0|} = 3|y|^{-1/3} \rightarrow \infty.$$

Consequently the usual local uniqueness hypothesis is unavailable at the initial point. The explicit examples prove nonuniqueness, even though every exhibited global solution has the same maximal interval $\mathbb{R}$.

Question 7

Let p be any real number. Consider the family of IVPs

$$y' = y(1 - y), \quad y(0) = p.$$

Focus on intervals of validity; a full equilibrium or phase-line analysis is not required.

Tasks

1. Derive a solution formula in terms of p , preserving the cases lost by division in separation.
2. Classify the maximal interval containing 0 for every real p . Give any finite endpoint explicitly.
3. Determine the direction and sign of blow-up whenever such an endpoint exists.
4. Verify the formula and explain why the interval depends on p even though the differential equation is smooth on the whole (x, y) -plane.

Question 7 – Solution

Strategy. A denominator zero is relevant only in the connected component containing the initial point. Preserve constant solutions before separating.

Step 1: Obtain a parameter formula. The constant solutions are $y = 0, 1$. Otherwise separation gives $\ln |y/(1-y)| = x + C$. Applying the initial value yields a formula that also includes both constants:

$$y(x) = \frac{pe^x}{1 - p + pe^x}.$$

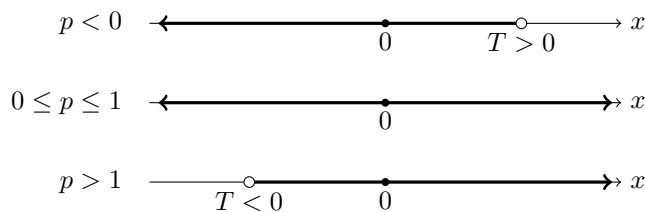
Its denominator $D(x)$ satisfies $D(0) = 1$. A zero requires $e^x = (p-1)/p > 0$, which occurs exactly when $p < 0$ or $p > 1$.

Step 2: Select the component containing zero. When it exists, set $T = \ln((p-1)/p)$. The complete classification is

initial value	endpoint location	maximal interval
$p < 0$	$T > 0$	$(-\infty, T)$
$0 \leq p \leq 1$	no finite endpoint	$\mathbb{R}$
$p > 1$	$T < 0$	(T, ∞)

For $0 < p < 1$, both terms of $D = 1 - p + pe^x$ are positive. The cases $p = 0, 1$ are global constants.

Step 3: Determine the endpoint behavior. If $p < 0$, then $D \downarrow 0^+$ as $x \uparrow T$, while $pe^x \rightarrow p - 1 < 0$, so $y \rightarrow -\infty$. If $p > 1$, then $D \downarrow 0^+$ as $x \downarrow T$, while $pe^x \rightarrow p - 1 > 0$, so $y \rightarrow +\infty$. Neither pole is removable.



Interval diagrams; endpoint positions schematic

Step 4: Verify and interpret. With $N = pe^x$, $N' = D' = N$, and

$$y' = \frac{N(D - N)}{D^2} = y(1 - y), \quad y(0) = p.$$

Smoothness gives local existence and uniqueness. It does not prevent the finite-time blow-up found for $p \notin [0, 1]$; the initial value determines whether the selected solution encounters a pole.

Question 8

Consider the equation in its original, undivided form:

$$xy' = 2y, \quad y(1) = 1.$$

It is defined for all real x , including $x = 0$. A classical solution must be continuously differentiable and satisfy this equation at every point of its open interval.

Tasks

1. Solve on $x > 0$ and explain why dividing by x cannot by itself settle continuation through 0 for the original equation.
2. Find every global classical solution of the original IVP.
3. Check both the differential equation and continuous differentiability at 0 for your full family.
4. Determine the maximal interval of each such solution. Explain how uniqueness on $x > 0$ can coexist with nonunique global continuation, and why this differs from an equation explicitly defined only for $x \neq 0$.

Question 8 – Solution

Strategy. The original equation remains meaningful at the vanishing leading coefficient. Solve on the two half-lines and test the join in that original equation.

Step 1: Solve where division is allowed. For $x \neq 0$, the equation gives $(y/x^2)' = 0$. The initial condition therefore forces $y = x^2$ throughout $x > 0$. On $x < 0$, the general solution is $y = Cx^2$, with an independent real constant C .

The divided equation $y' = 2y/x$ has no meaning at 0, but the original equation does: it requires $y(0) = 0$. A singularity introduced by division is not automatically an obstruction for the original problem.

Step 2: List all global solutions. Every global candidate must have the form

$$y_C(x) = \begin{cases} Cx^2, & x < 0, \\ 0, & x = 0, \\ x^2, & x > 0, \end{cases} \quad C \in \mathbb{R}.$$

The half-line solutions and the equation at 0 leave no other possible global classical solutions.

Step 3: Verify the join. For every finite C , y_C is continuous at 0. The difference quotient there is Ch for $h < 0$ and h for $h > 0$, so $y'_C(0) = 0$. Moreover,

$$y'_C(x) = 2Cx \quad (x < 0), \quad y'_C(x) = 2x \quad (x > 0)$$

tends to 0 from both sides. Thus y_C is continuously differentiable. The identity $xy'_C = 2y_C$ holds on each half-line and reduces to $0 = 0$ at 0. Also $y_C(1) = 1$.

Step 4: Resolve maximality and uniqueness. Each y_C has maximal interval $\boxed{\mathbb{R}}$. The unique positive-half-line solution extends through 0 in infinitely many ways because the original equation loses its derivative term there. The usual normal-form uniqueness theorem cannot cross that degeneracy.

If the problem had instead specified $y' = 2y/x$ only for $x \neq 0$, its IVP interval containing 1 would be $(0, \infty)$, even for the polynomial formula. Which equation and domain were actually given matters.

Question 9

Define a real function on the entire real line by

$$f(x) = \begin{cases} -1, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Consider $y' = f(x)$ with $y(-1) = 0$. Require a classical solution: y is continuously differentiable on an open interval and the equation holds at every point.

Tasks

1. Find the IVP solution on $x < 0$ and its maximal interval containing -1 .
2. Compute the finite left-hand limits of y and y' at 0.
3. Find the continuous function on $\mathbb{R}$ that agrees with this solution on $x < 0$ and solves the differential equation on $x > 0$. Decide whether it is a classical extension through 0.
4. Explain why this example does not contradict a continuation theorem requiring a continuous right-hand side. Would changing only $f(0)$ repair the obstruction?

Question 9 – Solution

Strategy. A formula obtained by integrating on either side of a jump need not be differentiable at the joining point.

Step 1: Solve on the initial component. For $x < 0$, $y' = -1$, so the initial condition gives

$$\boxed{y = -x - 1}.$$

It solves the equation at every negative x . Its maximal classical interval containing -1 is $\boxed{(-\infty, 0)}$, as the join test below shows.

Step 2: Compute the finite endpoint limits. As $x \uparrow 0$, $y(x) \rightarrow -1$ and $y'(x) \rightarrow -1$. Neither the solution nor its derivative blows up. The obstruction will be the change of slope imposed immediately to the right.

Step 3: Test the unique continuous gluing. On $x > 0$, any solution is $x + C$. Continuity with the left branch forces $C = -1$ and the value $y(0) = -1$. The only continuous gluing is therefore

$$Y(x) = |x| - 1.$$

Its left difference quotient at 0 is -1 and its right difference quotient is 1 . Consequently $Y'(0)$ does not exist. It is not a classical solution of the equation on any interval containing 0.

Any larger interval extending the negative-half-line solution would include points on both sides of 0 and would require this same gluing. Thus no classical extension is possible, proving the claimed maximality despite the finite limits.

Step 4: Identify the missing hypothesis. Here f is defined everywhere but is discontinuous at 0. A continuation result that requires continuity of the right-hand side near the endpoint cannot be applied. Domain membership and bounded values alone are insufficient when that hypothesis fails.

Changing $f(0)$ to any single real value cannot reconcile the unequal one-sided derivatives forced by $f = -1$ to the left and $f = 1$ to the right. The same corner remains. The function Y solves the equation away from the join, but the question requires the equation at every point of a classical interval.

Question 10

Design an initial-value problem of the form

$$y' = (Ax + B)y^2, \quad y(0) = 1,$$

where A, B are real constants, so that its solution has exactly the maximal interval $(-2, 3)$ and tends to $+\infty$ at both finite endpoints.

Tasks

1. Use the reciprocal variable $u = 1/y$ to determine A and B from the prescribed endpoints.
2. Recover the solution and prove that it is positive and has exactly the requested maximal interval.
3. Verify the IVP and prove that your coefficient pair is the only affine choice with these properties.
4. Find the solution's minimum on its interval and sketch the curve with its endpoint asymptotes. Explain why smooth coefficients do not force a global solution here.

Question 10 – Solution

Strategy. For a quadratic-growth equation, the reciprocal solution is a quadratic polynomial. Put its zeros at the desired endpoints.

Step 1: Determine the reciprocal polynomial. While $y \neq 0$,

$$u' = -(Ax + B), \quad u(0) = 1, \quad u = 1 - Bx - \frac{A}{2}x^2.$$

Blow-up at both prescribed endpoints requires $u(-2) = u(3) = 0$. A quadratic with these roots and value 1 at 0 must be

$$u = \frac{(x+2)(3-x)}{6} = 1 + \frac{x}{6} - \frac{x^2}{6}.$$

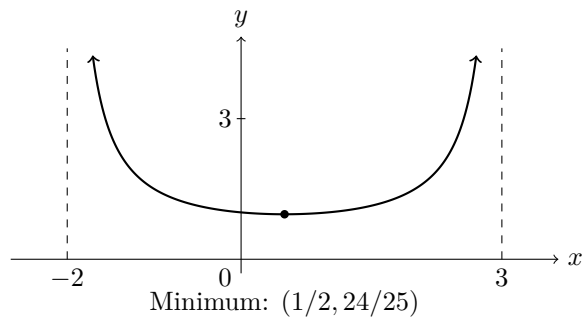
Thus $A = 1/3$, $B = -1/6$ and

$$y = \frac{6}{(x+2)(3-x)}, \quad I = (-2, 3).$$

Step 2: Check the interval and uniqueness of the design. Both denominator factors are positive on I . At either endpoint the denominator tends to 0^+ , so $y \rightarrow +\infty$ and no finite continuous extension exists.

Also $y' = -u'/u^2 = ((2x-1)/6)y^2$ and $y(0) = 1$. For any proposed affine pair, a solution through 1 cannot cross the zero solution, by local uniqueness for the smooth right-hand side. Hence its reciprocal on the proposed interval has exactly the polynomial form above. The two endpoint zeros and $u(0) = 1$ determine it uniquely, proving uniqueness of the coefficient pair.

Step 3: Locate the minimum. The denominator is $25/4 - (x - 1/2)^2$, so it is largest at $x = 1/2$. Therefore $y_{\min} = 24/25$ there. Equivalently, y' changes from negative to positive at $1/2$.



Step 4: Interpret the construction. The coefficients and the right-hand side are smooth everywhere, but the solution grows without bound in both finite endpoint directions. Smoothness ensures local solvability and uniqueness, not global existence for this nonlinear equation.