

Question 1

Consider the homogeneous first-order equation

$$y' = \frac{y}{x} + \sqrt{1 + (y/x)^2}, \quad y(1) = 0, \quad x > 0.$$

Here “homogeneous” means that the right-hand side depends only on y/x ; it does not mean a homogeneous linear equation.

Tasks

1. Set $v = y/x$ and derive the equation for v , showing the product-rule term in y' .
2. Solve the transformed IVP and recover an explicit formula for y .
3. Verify the original equation, taking care with the nonnegative square root, and find the maximal interval within $x > 0$.
4. A student uses $y' = xv'$ when $y = xv$. Identify the omitted term and explain why that error changes the transformed equation.

Question 1 – Solution

Strategy. The ratio substitution removes the repeated y/x , but differentiating $y = xv(x)$ requires both product-rule terms.

Step 1: Transform the equation. Since $y' = v + xv'$, substitution gives

$$v + xv' = v + \sqrt{1 + v^2}, \quad \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}.$$

The initial value is $v(1) = 0$. An antiderivative on the left is $\ln(v + \sqrt{1 + v^2})$, whose logarithm has a strictly positive argument for every real v . Hence

$$\ln(v + \sqrt{1 + v^2}) = \ln x.$$

Step 2: Recover the original unknown. Exponentiating gives $v + \sqrt{1 + v^2} = x$. Squaring $\sqrt{1 + v^2} = x - v$ and simplifying gives

$$v = \frac{x^2 - 1}{2x}, \quad \boxed{y = \frac{x^2 - 1}{2}}.$$

This step is valid: for the recovered expression, $x - v = (x^2 + 1)/(2x) > 0$ on $x > 0$, so the unsquared relation also holds.

Step 3: Verify the branch and interval. For this y ,

$$\sqrt{1 + (y/x)^2} = \frac{x^2 + 1}{2x}, \quad \frac{y}{x} + \sqrt{1 + (y/x)^2} = x = y'.$$

Also $y(1) = 0$. The formula satisfies the original equation for every $x > 0$, so its maximal interval there is $\boxed{(0, \infty)}$. The polynomial has a value at 0, but the given equation does not; an algebraic extension alone does not extend this solution through the excluded point.

Step 4: Diagnose the missing derivative term. The omitted term is v . Writing only xv' would produce $xv' = v + \sqrt{1 + v^2}$ instead of $xv' = \sqrt{1 + v^2}$. The two equations describe different ratio functions, so the product rule is essential to the reduction.

Question 2

On $x > 0$, consider

$$y' = \frac{y}{x} + (y/x - 1)(y/x - 2), \quad y(1) = 3.$$

Tasks

1. Use $v = y/x$ to derive a separable equation. Find every constant value of v before dividing by a factor involving v .
2. Solve the transformed equation, and give the resulting original solution family together with any missing straight-line solution.
3. Select the IVP solution and its maximal interval containing 1.
4. Verify the IVP solution and explain why it cannot cross either straight-line solution within that interval.

Question 2 – Solution

Strategy. Constant ratios correspond to straight lines, so preserve them before separating the nonconstant ratio equation.

Step 1: Transform and preserve the constants. Writing $y = xv$ gives $xv' = (v - 1)(v - 2)$. The constant ratios $v = 1, 2$ yield the solutions $y = x, y = 2x$. For other ratios,

$$\int \frac{dv}{(v-1)(v-2)} = \int \frac{dx}{x}, \quad \ln \left| \frac{v-2}{v-1} \right| = \ln x + C_0.$$

Thus $(v - 2)/(v - 1) = Cx$, with nonzero C for nonconstant v , and

$$y = \frac{x(2 - Cx)}{1 - Cx}.$$

Allowing $C = 0$ restores $y = 2x$; $y = x$ remains a separate solution. Each formula is used only on intervals where its denominator is nonzero.

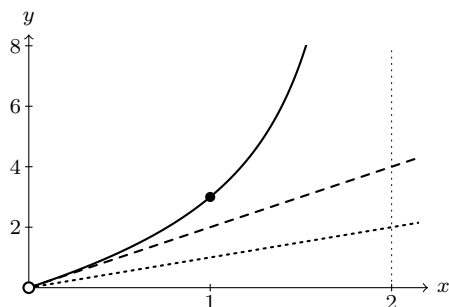
Step 2: Select the IVP branch. The condition $v(1) = 3$ gives $C = 1/2$, hence

$$y = \frac{x(2 - x/2)}{1 - x/2}, \quad I = (0, 2).$$

The coefficient domain excludes 0, while $y \rightarrow +\infty$ as $x \uparrow 2$.

Step 3: Verify and compare. For $v = (2 - Cx)/(1 - Cx)$, $v' = C/(1 - Cx)^2$, and $(v - 1)(v - 2) = Cx/(1 - Cx)^2 = xv'$. Thus $y' = v + xv'$ recovers the original equation, and the selected initial value is 3.

On $0 < x < 2$, $v = 1 + 1/(1 - x/2) > 2$. Therefore $y > 2x > x$ throughout the interval: neither line is crossed.



Solid: IVP solution. Dashed: $y = 2x$. Dotted sloping line: $y = x$. The domain is $x > 0$.

Question 3

Consider an IVP on the negative half-line:

$$y' = \frac{y}{x} + \frac{x}{y}, \quad y(-1) = 1, \quad x < 0,$$

with the original equation defined only when $y \neq 0$.

Tasks

1. Use $v = y/x$ and separate the equation, keeping the correct logarithmic absolute value.
2. Select both the constant and the sign of the transformed solution from the initial condition.
3. Recover y and determine its maximal interval containing -1 within $x < 0$.
4. Verify the equation and describe the finite endpoint. Explain why replacing $\ln|x|$ by $\ln x$, or choosing the positive sign for v , would fail here.

Question 3 – Solution

Strategy. The substitution is valid for negative x too, but its initial ratio is negative and the logarithm must respect the domain.

Step 1: Transform and integrate. From $y = xv$ and $y' = v + xv'$,

$$xv' = \frac{1}{v}, \quad v \, dv = \frac{dx}{x}, \quad v^2 = 2 \ln |x| + C.$$

At $x = -1$, $v = y/x = -1$, so $C = 1$. Continuity and the nonzero ratio select

$$v = -\sqrt{1 + 2 \ln |x|}.$$

Step 2: Recover the branch and domain. The radicand must be strictly positive, since zero would give the excluded value $y = 0$. Thus $|x| > e^{-1/2}$. On the negative half-line this gives

$$\boxed{y = -x\sqrt{1 + 2 \ln |x|}, \quad I = (-\infty, -e^{-1/2})}.$$

It is positive and gives $y(-1) = 1$.

Step 3: Verify and check the endpoint. Let $D = 1 + 2 \ln |x|$. Since $D' = 2/x$,

$$y' = -\sqrt{D} - \frac{1}{\sqrt{D}}, \quad \frac{y}{x} + \frac{x}{y} = -\sqrt{D} - \frac{1}{\sqrt{D}}.$$

At the finite endpoint approached from the left, $D \downarrow 0$, so $y \rightarrow 0$ from above and $y' \rightarrow -\infty$. The original equation is undefined there, and no differentiable extension through it is possible.

Step 4: Identify the two domain mistakes. The real logarithm $\ln x$ is not defined for these negative x ; the antiderivative is $\ln |x|$. Also $v > 0$ would give $y = xv < 0$ on this half-line, contradicting $y(-1) = 1$. The sign must be selected for the transformed ratio, not guessed from the sign of y alone.

Question 4

Consider the nonseparable-looking equation

$$y' = (x + y)^2 - 1, \quad y(0) = 1.$$

Tasks

1. Set $u = x + y$. Derive its differential equation, including the derivative of x .
2. Solve the transformed IVP, recover y , and find its maximal interval containing 0.
3. Determine the minimum of the selected solution and verify the original equation.
4. Find the original solution represented by the constant transformed value $u = 0$. Explain why dividing by u^2 would miss it and why it is not a constant function y .

Question 4 – Solution

Strategy. Replace the repeated linear combination by one variable. The extra derivative of x cancels the constant term in the equation.

Step 1: Reduce to a separable IVP. For $u = x + y$, $u' = 1 + y' = u^2$, and $u(0) = 1$. Separating on the nonzero branch gives

$$-\frac{1}{u} = x + C, \quad C = -1, \quad u = \frac{1}{1-x}.$$

Thus

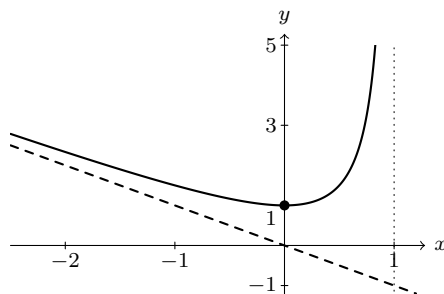
$$y = -x + \frac{1}{1-x}, \quad I = (-\infty, 1).$$

The only finite obstruction is the pole at $x = 1$, where $y \rightarrow +\infty$ from the left.

Step 2: Find the minimum and verify. Differentiation gives $y' = -1 + (1-x)^{-2}$. On I this is negative for $x < 0$, zero at 0, and positive for $0 < x < 1$. Hence the unique global minimum on I is $y(0) = 1$.

Also $x + y = 1/(1-x)$, so $(x + y)^2 - 1 = (1-x)^{-2} - 1 = y'$, and the initial value checks directly.

Step 3: Restore the constant transformed branch. The equation $u' = u^2$ also has $u \equiv 0$. In the original variables it gives $y = -x$ on all of $\mathbb{R}$. Its derivative is -1 and its right-hand side is $0^2 - 1 = -1$, so it is valid. Division by u^2 excludes it. A constant value of a combination of x and y need not represent a constant value of y .



Solid: IVP solution, with minimum $(0, 1)$. Dashed: the restored solution $y = -x$.

Question 5

Consider

$$y' = \frac{x + y - 3}{x - y + 1}, \quad y(2) = 2,$$

on the domain $x - y + 1 \neq 0$.

Tasks

1. Find constants h, k such that $X = x - h, Y = y - k$ remove the constant terms in both affine expressions. Write the shifted equation and initial point.
2. On a neighborhood of that point with $X > 0$, use $v = Y/X$ to obtain an implicit solution in X, Y .
3. Verify the local implicit relation and find the slope at the original initial point.
4. Is $X = 0$ automatically a singular line of the original differential equation? Identify the actual excluded line and explain the difference between these restrictions.

Question 5 – Solution

Strategy. Translate the intersection of the two affine lines to the origin, then apply the homogeneous ratio substitution locally.

Step 1: Choose the translation. The equations $h + k - 3 = 0$ and $h - k + 1 = 0$ give $\boxed{h = 1, k = 2}$. Hence

$$X = x - 1, \quad Y = y - 2, \quad \frac{dY}{dX} = \frac{X + Y}{X - Y}, \quad (X, Y) = (1, 0) \text{ initially.}$$

A translation does not change the derivative because $dX/dx = 1$.

Step 2: Substitute and integrate. With $Y = Xv$,

$$v + Xv' = \frac{1 + v}{1 - v}, \quad \frac{1 - v}{1 + v^2} dv = \frac{dX}{X}.$$

On the local region $X > 0$, integration gives

$$\arctan v - \frac{1}{2} \ln(1 + v^2) = \ln X + C.$$

At $(1, 0)$, $C = 0$. Substituting $v = Y/X$ and simplifying yields

$$\boxed{\arctan(Y/X) - \frac{1}{2} \ln(X^2 + Y^2) = 0}$$

for the local branch through $(1, 0)$ with $X > 0$.

Step 3: Verify and find the slope. For the displayed left side F , differentiation gives

$$F_X = -\frac{X + Y}{X^2 + Y^2}, \quad F_Y = \frac{X - Y}{X^2 + Y^2}.$$

Thus $F_X + F_Y Y' = 0$ recovers the shifted equation wherever $X - Y \neq 0$. At $(1, 0)$, $F_Y = 1$, so a local graph exists and $\boxed{y'(2) = 1}$.

Step 4: Separate coordinate and equation restrictions. The ratio Y/X excludes $X = 0$, namely $x = 1$, but the original equation is defined there whenever $y \neq 2$. Its actual excluded line is $\boxed{x - y + 1 = 0}$, or $y = x + 1$. An alternative local coordinate or angle branch can be needed when the ratio chart fails; that failure alone does not prove that an original solution ends. No global branch claim is made by this local arctangent formula.

Question 6

Consider an equation with parallel affine expressions:

$$y' = \frac{2x + y + 1}{4x + 2y + 3}, \quad y(0) = 0,$$

where $4x + 2y + 3 \neq 0$.

Tasks

1. Explain why no translation of x, y removes the constants in both numerator and denominator simultaneously.
2. Use $u = 2x + y + 1$ to derive a separable equation, then obtain an implicit formula for the IVP.
3. Find the solution corresponding to a constant u that would be lost during separation, and verify it in the original equation.
4. Prove that the selected implicit formula defines the IVP solution for every real x , while remaining inside the original coefficient domain.

Question 6 – Solution

Strategy. When the affine forms cannot be centered together, use their shared linear combination directly.

Step 1: Reject a simultaneous centering. A translation by (h, k) would require $2h + k + 1 = 0$ and $4h + 2k + 3 = 0$. Doubling the first equation gives $4h + 2k + 2 = 0$, contradicting the second. Their zero lines are parallel and distinct.

Step 2: Reduce and integrate. Set $u = 2x + y + 1$. Then the denominator is $2u + 1$, and

$$u' = 2 + y' = \frac{5u + 2}{2u + 1}, \quad \frac{2u + 1}{5u + 2} du = dx.$$

The original domain excludes $u = -1/2$. For $u \neq -2/5$, an antiderivative is

$$H(u) = \frac{2}{5}u + \frac{1}{25} \ln |5u + 2|.$$

Since $u(0) = 1$, the IVP relation is

$$H(u) = x + \frac{2}{5} + \frac{1}{25} \ln 7, \quad y = u - 2x - 1.$$

Step 3: Restore the lost constant combination. The transformed equation admits $u \equiv -2/5$, giving $y = -2x - 7/5$. Its numerator is $-2/5$ and its denominator is $1/5$, so the right side is $-2 = y'$. It is a global solution, but its initial value is not 0.

Step 4: Prove the selected branch is global. The initial value lies in $u > -2/5$. On this interval,

$$H'(u) = \frac{2u + 1}{5u + 2} > 0,$$

and $H(u)$ tends to $-\infty$ at the left endpoint and to $+\infty$ as $u \rightarrow \infty$. Thus the implicit relation gives exactly one differentiable u for every real x . Also $2u + 1 > 1/5$, so the original denominator never vanishes. Differentiating $H(u) = x + H(1)$ gives the transformed equation and then the original equation; this verifies the unique global IVP branch.

Question 7

Consider

$$y' = xe^{-y} - 1, \quad y(0) = 0.$$

Tasks

1. Set $u = e^y$ and derive a linear differential equation for u .
2. Solve it using the initial condition and recover y .
3. Prove that the selected transformed solution stays positive for every real x . Find the minimum value of y and where it occurs.
4. The same linear equation has the solution $u = x - 1$. Determine precisely where it produces a real original solution, and explain why a valid transformed solution need not be invertible everywhere.

Question 7 – Solution

Strategy. Exponentiating the unknown makes the reciprocal exponential linear, but the inverse logarithm requires a positive transformed value.

Step 1: Transform and solve. For $u = e^y$, $u' = e^y y' = x - u$. The initial value is $u(0) = 1$, so

$$u' + u = x, \quad (e^x u)' = x e^x,$$

which gives $u = x - 1 + C e^{-x}$. The data select $C = 2$, hence

$$y(x) = \ln(x - 1 + 2e^{-x}).$$

Step 2: Establish the inverse domain and minimum. Let $U(x) = x - 1 + 2e^{-x}$. Then $U' = 1 - 2e^{-x}$ and $U'' = 2e^{-x} > 0$. Its unique global minimum occurs at $x = \ln 2$ and equals $\ln 2 > 0$. Therefore the logarithm is defined and smooth for all real x , so the IVP interval is $\boxed{\mathbb{R}}$.

Since the logarithm is increasing, the solution has unique minimum

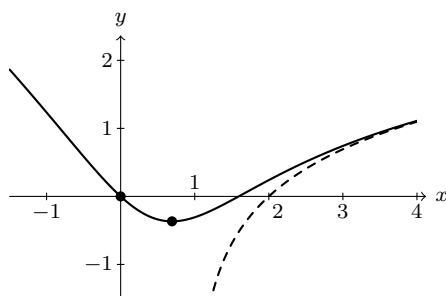
$$y(\ln 2) = \ln(\ln 2).$$

Verification is immediate from $U' = x - U$: $y' = U'/U = x/U - 1 = x e^{-y} - 1$, and $y(0) = \ln 1 = 0$.

Step 3: Test another transformed solution. The linear solution $u = x - 1$ is defined everywhere, but $u > 0$ only for $x > 1$. It gives

$$y = \ln(x - 1), \quad x > 1.$$

On that interval, $y' = 1/(x - 1) = x/(x - 1) - 1$, verifying the original equation. At $x \downarrow 1$, $y \rightarrow -\infty$, and for $x \leq 1$ no real logarithm of u exists. The map $y \mapsto e^y$ has range $(0, \infty)$; solving the transformed linear equation does not remove that range restriction.



Solid: the global IVP solution. Dashed: $\ln(x - 1)$, which is valid only for $x > 1$.

Question 8

On $x > 0$, consider

$$x^2 y' + y = e^{-1/x}.$$

We seek a solution that remains bounded as $x \downarrow 0$.

Tasks

1. Change the independent variable to $t = 1/x$ and define $Y(t) = y(1/t)$. Derive the transformed equation carefully using the chain rule.
2. Solve the transformed equation and recover the full original solution family on $x > 0$.
3. Select the unique solution bounded as $x \downarrow 0$. Find its limit at 0, its limit as $x \rightarrow \infty$, and its value at $x = 1$.
4. Verify it in the original equation and explain how the substitution changes the endpoint at which the boundedness condition is imposed.

Question 8 – Solution

Strategy. A reciprocal change of the independent variable absorbs x^2 , while converting a finite endpoint into an infinite one.

Step 1: Transform the derivative. Since $t = 1/x$, $dt/dx = -1/x^2$. Writing $y(x) = Y(t)$ gives

$$x^2 y' = -Y'(t), \quad -Y' + Y = e^{-t}.$$

Thus $\boxed{Y' - Y = -e^{-t}}$ for $t > 0$. The minus sign comes from the reversal of the independent variable.

Step 2: Solve and return to x . The integrating factor is e^{-t} :

$$(e^{-t}Y)' = -e^{-2t}, \quad Y = \frac{1}{2}e^{-t} + Ce^t.$$

Therefore all solutions on the stated half-line are

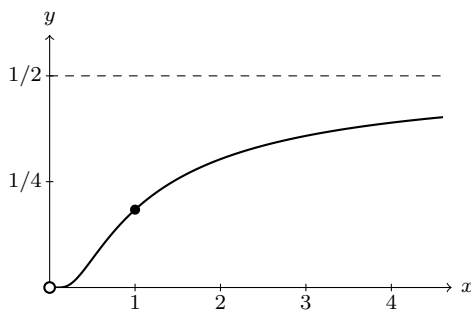
$$\boxed{y(x) = \frac{1}{2}e^{-1/x} + Ce^{1/x}}.$$

Step 3: Apply the endpoint condition. As $x \downarrow 0$, $t \rightarrow +\infty$. The first term tends to zero, whereas any nonzero C makes the second term unbounded. Consequently the unique bounded choice is

$$\boxed{y_b(x) = \frac{1}{2}e^{-1/x}}.$$

Its limits are 0 as $x \downarrow 0$ and $1/2$ as $x \rightarrow \infty$, and $\boxed{y_b(1) = 1/(2e)}$.

Step 4: Verify and interpret the reversal. Differentiation gives $y'_b = e^{-1/x}/(2x^2)$, so $x^2 y'_b + y_b = e^{-1/x}$. The condition near $x = 0$ becomes a condition at $t = \infty$, not at $t = 0$. Conversely, $x \rightarrow \infty$ corresponds to $t \downarrow 0$. The limit at $x = 0$ does not make that point part of the stated domain of the equation.



The bounded solution approaches the excluded point $(0, 0)$ and tends to $1/2$ as $x \rightarrow \infty$.

Question 9

Consider the equation

$$y' = y^2 + (1 - 2x)y + x^2 - x + 1, \quad y(0) = 2.$$

A particular solution is suggested: $y_p(x) = x$.

Tasks

1. Verify the suggested solution and derive the equation for $z = y - y_p$.
2. On a branch with $z \neq 0$, set $u = 1/z$ and derive a linear equation for u .
3. Solve the IVP and find its maximal interval containing 0.
4. Verify the recovered solution through the substitutions, restore the solution excluded by $u = 1/z$, and explain why this reciprocal is applied to the difference from y_p rather than directly to y .

Question 9 – Solution

Strategy. Subtract a known particular solution so that the constant term cancels, then use a reciprocal on the remaining quadratic equation.

Step 1: Verify and subtract the particular solution. For $y_p = x$, the right-hand side is $x^2 + (1 - 2x)x + x^2 - x + 1 = 1 = y'_p$. Now write $y = x + z$. Substitution and cancellation give

$$1 + z' = 1 + z + z^2, \quad \boxed{z' = z + z^2}, \quad z(0) = 2.$$

Step 2: Make the remaining equation linear. For $z \neq 0$, $u = 1/z$ gives

$$u' = -\frac{z'}{z^2} = -u - 1, \quad u(0) = \frac{1}{2}.$$

Solving $u' + u = -1$ yields $u = Ce^{-x} - 1$, and the initial value requires $C = 3/2$.

Step 3: Recover the IVP and its interval. Thus

$$\boxed{y = x + \frac{1}{\frac{3}{2}e^{-x} - 1}, \quad I = (-\infty, \ln(3/2))}.$$

The denominator is positive before its unique zero at $\ln(3/2)$ and gives $y(0) = 2$. At that endpoint $y \rightarrow +\infty$, preventing a finite differentiable extension.

Step 4: Verify and account for the missing solution. Where $u \neq 0$, $z = 1/u$ satisfies $z' = -u'/u^2 = z + z^2$. Thus $y' = 1 + z + z^2$, which is exactly the original right side after $y = x + z$. The excluded case $z \equiv 0$ restores $\boxed{y = x}$, a global solution with a different initial value.

The original equation contains a nonzero term independent of y . A direct reciprocal $w = 1/y$ would retain a quadratic term in w and would not be linear. Subtracting the known solution first cancels that obstruction. This two-step substitution is the useful structure, not the reciprocal alone.

Question 10

For a homogeneous equation $y' = F(y/x)$ on $x > 0$, introduce the logarithmic independent variable $t = \ln x$ and the ratio $v(t) = y(e^t)/e^t$.

We want F to be a polynomial of degree at most two such that, in the transformed equation, $v = 1$ and $v = 3$ are equilibrium values and $dv/dt = -1$ when $v = 2$.

Tasks

1. Derive the equation for dv/dt in terms of F and v .
2. Determine the unique polynomial F satisfying the design requirements.
3. Solve the resulting original IVP with $y(1) = 2$, showing the transformed integration and recovery of y .
4. Find the limits of y/x as $x \downarrow 0$ and as $x \rightarrow \infty$. Identify the original solutions represented by the equilibrium ratios and explain why they are not constant functions y . Decide whether the selected original $y(x)$ increases or decreases.

Question 10 – Solution

Strategy. The logarithmic independent variable turns multiplication of x into translation of t and removes the factor x from the ratio equation.

Step 1: Derive and design the transformed equation. Writing $y = xv(\ln x)$ gives $y' = v + dv/dt$, so

$$\frac{dv}{dt} = F(v) - v.$$

This polynomial has roots 1, 3 and degree at most two, hence equals $k(v-1)(v-3)$. Its value -1 at $v = 2$ forces $k = 1$. Therefore

$$F(v) = v^2 - 3v + 3, \quad v' = (v-1)(v-3).$$

Step 2: Solve the selected trajectory. The data give $v(0) = 2$, between the equilibria. Separation yields

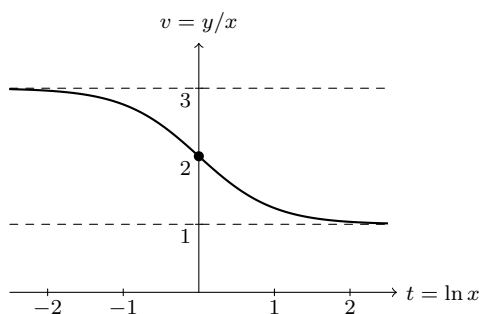
$$\frac{1}{2} \ln \left| \frac{v-3}{v-1} \right| = t + C, \quad C = 0.$$

The ratio is negative on the selected branch, so $(v-3)/(v-1) = -e^{2t}$. Thus

$$v = 1 + \frac{2}{1 + e^{2t}}, \quad y = x + \frac{2x}{1 + x^2}, \quad x > 0.$$

Directly $dv/dt = -4e^{2t}/(1 + e^{2t})^2 = (v-1)(v-3)$, so the chain rule verifies the original equation and $y(1) = 2$.

Step 3: Interpret the limits and equilibria. As $x \downarrow 0$, $t \rightarrow -\infty$ and $y/x \rightarrow 3$; as $x \rightarrow \infty$, $t \rightarrow \infty$ and $y/x \rightarrow 1$. The constant ratios give $y = x$, $y = 3x$, both valid on $x > 0$. They are straight lines, not constant original functions: constancy applies to the ratio in the new coordinates. Moreover, $y' = F(v) = (v-3/2)^2 + 3/4 > 0$, so the original y increases even though its ratio v decreases.



The transformed ratio decreases from 3 to 1; the dashed levels represent $y = 3x$ and $y = x$.