

Question 1

A Bernoulli equation has the form $y' + p(x)y = q(x)y^n$. Consider the positive initial-value problem

$$y' + \frac{2}{x}y = x^3y^2, \quad y(1) = 1, \quad x > 0.$$

Tasks

1. For positive y and $n \neq 1$, derive the linear equation satisfied by $v = y^{1-n}$ using the chain rule.
2. Apply that reduction to the given IVP and solve the resulting linear equation with an integrating factor.
3. Find the maximal open interval containing 1, and determine the minimum of the selected solution on that interval.
4. Verify the answer in the original equation and identify a solution omitted by the substitution.

Question 1 – Solution

Strategy. Derive the reduction first, then retain only the interval where its inverse gives the selected positive solution.

Step 1: Reduce to a linear equation. Multiplying by y^{-n} and using $v' = (1 - n)y^{-n}y'$ gives

$$v' + (1 - n)pv = (1 - n)q.$$

Here $n = 2$, so $v = 1/y$ obeys $v' - 2v/x = -x^3$. The integrating factor is x^{-2} , hence

$$(x^{-2}v)' = -x, \quad v = Cx^2 - \frac{x^4}{2}.$$

Since $v(1) = 1$, $C = 3/2$.

Step 2: Invert and analyze. Thus

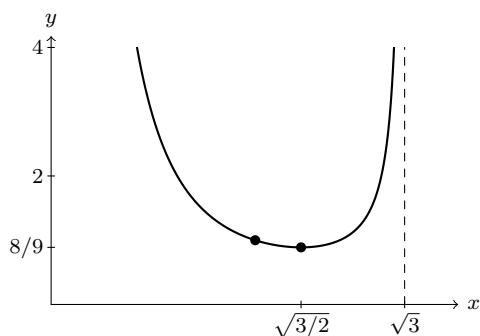
$$y = \frac{2}{x^2(3 - x^2)}, \quad I = (0, \sqrt{3}).$$

The denominator $3x^2 - x^4$ is positive on I , vanishes at its endpoints, and has its unique maximum $9/4$ at $x = \sqrt{3/2}$. The solution therefore has minimum $8/9$ there and tends to $+\infty$ at both ends of I .

Step 3: Verify and restore the lost solution. With $D = 3x^2 - x^4$, $y' = -2(6x - 4x^3)/D^2$, so

$$y' + 2y/x = 4x^3/D^2 = x^3y^2, \quad y(1) = 1.$$

Also $y \equiv 0$ satisfies the original equation on $x > 0$ but is excluded by $v = 1/y$; it does not satisfy this IVP.



The selected branch has minimum $8/9$ and diverges at both interval endpoints.

Question 2

For the IVP

$$y' + y = e^{2x}y^3, \quad y(0) = -1,$$

a student sets $v = y^{-2}$ but writes $v' + 2v = 2e^{2x}$ and then plans to use $y = 1/\sqrt{v}$.

Tasks

1. Find both errors in this procedure and derive the correct equation for v .
2. Solve the linear equation and select the original solution using the initial condition.
3. Find its maximal interval containing 0, explaining why a positive transformed variable does not determine the sign of y .
4. Verify the original IVP, and state whether the opposite sign and the zero function solve the same differential equation.

Question 2 – Solution

Strategy. The derivative of y^{-2} carries a negative factor, and the inverse transformation needs a separately chosen sign.

Step 1: Correct the transformed equation. For $y \neq 0$, divide by y^3 and use $v' = -2y^{-3}y'$. This gives

$$-\frac{1}{2}v' + v = e^{2x}, \quad \boxed{v' - 2v = -2e^{2x}}.$$

The student's two coefficient signs are reversed. Moreover, $v = 1/y^2$ is the same for y and $-y$, so selecting the positive inverse would contradict the negative initial value.

Step 2: Solve and choose the branch. The integrating factor is e^{-2x} :

$$(e^{-2x}v)' = -2, \quad v = e^{2x}(C - 2x).$$

The condition $v(0) = 1$ gives $C = 1$. Continuity and $y(0) = -1$ select

$$\boxed{y = -\frac{e^{-x}}{\sqrt{1-2x}}, \quad I = (-\infty, 1/2)}.$$

Here $v > 0$. At $x \uparrow 1/2$, the solution tends to $-\infty$, so there is no finite extension; for $x > 1/2$, this v cannot equal $1/y^2$ for real y .

Step 3: Verify and distinguish other solutions. On I , logarithmic differentiation of $|y|$ gives

$$\frac{y'}{y} = -1 + \frac{1}{1-2x}.$$

Consequently $y' + y = y/(1-2x) = e^{2x}y^3$, and the initial value is -1 . The opposite branch $+e^{-x}/\sqrt{1-2x}$ also solves the differential equation, with initial value 1 . The zero function solves it as well but is omitted by y^{-2} . Neither of those alternatives solves the stated IVP.

Question 3

Seek nonnegative, continuously differentiable solutions for $t \geq 0$ of

$$y' + 2y = 2\sqrt{y}, \quad y(0) = 0,$$

where the prime denotes d/dt and $\sqrt{y}$ is the nonnegative square root.

Tasks

1. On an interval where $y > 0$, derive and solve the equation for $w = \sqrt{y}$.
2. Find every solution that waits at zero until a time $b \geq 0$ and then becomes positive. Include the possibility of staying zero forever.
3. Check differentiability and the original equation at the joining time, and justify that your list is exhaustive for $t \geq 0$.
4. Explain why squaring a linear formula with $w < 0$ does not give a valid solution of the original equation on that interval.

Question 3 – Solution

Strategy. The linearized equation applies only on positive pieces; restore zero intervals and enforce the sign of the square root.

Step 1: Solve a positive piece. Writing $y = w^2$ with $w > 0$ gives $2ww' + 2w^2 = 2w$, hence $w' + w = 1$. Thus $w = 1 + Ce^{-t}$. If positivity starts at b with $w(b) = 0$, then $w = 1 - e^{-(t-b)}$, positive precisely for $t > b$.

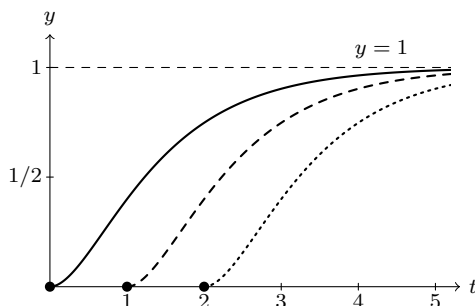
Step 2: Restore the zero intervals. For each finite $b \geq 0$, define

$$y_b(t) = \begin{cases} 0, & 0 \leq t \leq b, \\ (1 - e^{-(t-b)})^2, & t \geq b. \end{cases}$$

Also include $y \equiv 0$, denoted by $b = \infty$. On the positive piece let $r = e^{-(t-b)}$; then $y' = 2r(1 - r)$ and $y' + 2y = 2(1 - r) = 2\sqrt{y}$. At a positive joining time, both one-sided derivatives are 0; if $b = 0$, the right derivative is 0. Thus the solution is C^1 on its stated domain and satisfies the equation at the join.

Step 3: Prove completeness and reject a false extension. Any positive component must start from zero at its finite left endpoint, including $b = 0$, and the linear formula above fixes that component uniquely. It stays positive forever after b , so it cannot end at another zero. Hence there is at most one positive component and the list is exhaustive.

For $t < b$, the same linear expression has $w < 0$. If it is squared, $\sqrt{w^2} = |w| = -w$, while the computed left side is $2w$. Thus it fails the original equation there. Squaring cannot remove this sign restriction.



Solid: $b = 0$. Dashed: $b = 1$. Dotted: $b = 2$. Each is zero before its departure.

Question 4

Consider the negative-power Bernoulli equation

$$y' + y = \frac{e^{-x}}{y}, \quad y(0) = -1.$$

The original equation is defined only when $y \neq 0$.

Tasks

1. Derive the linear equation for $v = y^2$ and solve it using the initial condition.
2. Select the real branch for y and find its maximal interval containing 0.
3. Determine what happens to y and y' at its finite endpoint. Decide whether reaching the value zero permits continuation.
4. Verify the equation, and explain why the zero function cannot be restored as an omitted solution in this example.

Question 4 – Solution

Strategy. The squared variable must be strictly positive, and the original reciprocal term remains undefined at zero.

Step 1: Linearize and integrate. Multiplication by $2y$ yields $v' + 2v = 2e^{-x}$, where $v = y^2$. Using the integrating factor e^{2x} ,

$$(e^{2x}v)' = 2e^x, \quad v = 2e^{-x} + Ce^{-2x}.$$

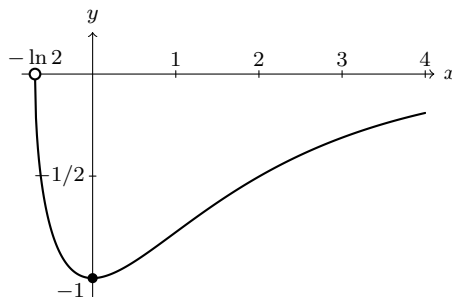
Since $v(0) = 1$, $C = -1$.

Step 2: Invert on the correct interval. The condition $v = e^{-2x}(2e^x - 1) > 0$ requires $x > -\ln 2$. The initial sign gives

$$y = -e^{-x}\sqrt{2e^x - 1}, \quad I = (-\ln 2, \infty).$$

As $x \downarrow -\ln 2$, $y \rightarrow 0$ from below. The equation gives $y' = e^{-x}/y - y \rightarrow -\infty$, so no differentiable extension exists at that endpoint. Also the original right-hand side is undefined there. As $x \rightarrow \infty$, $y \rightarrow 0$ from below, but it never reaches zero at a finite point of I .

Step 3: Verify and check the excluded value. Differentiating $y^2 = v$ gives $2yy' = v'$. Since $v' + 2v = 2e^{-x}$ and $y \neq 0$ on I , division by $2y$ recovers $y' + y = e^{-x}/y$. The selected formula gives $y(0) = -1$. Unlike positive-power examples, $y \equiv 0$ is not a solution: the original equation has no value at zero.



The finite endpoint is excluded; the negative branch approaches zero again as $x \rightarrow \infty$.

Question 5

For each real initial value a , consider forward solutions of

$$y' + y = y^3, \quad y(0) = a, \quad t \geq 0,$$

where the prime denotes d/dt .

Tasks

1. Solve using $v = y^{-2}$ when $a \neq 0$, retaining the sign of a . Treat $a = 0$ separately.
2. Classify all a according to whether the forward solution tends to zero, is constant, or blows up in finite time.
3. In the blow-up cases, find the exact lifetime and the sign of the divergence.
4. Verify the solution and explain why the threshold initial values are not included in either neighboring regime.

Question 5 – Solution

Strategy. A linear denominator exposes the threshold, while the sign of the original solution must be carried separately.

Step 1: Solve the transformed IVP. For $a \neq 0$, $v = y^{-2}$ gives $v' - 2v = -2$, so

$$v = 1 + (a^{-2} - 1)e^{2t}, \quad y = \frac{a}{\sqrt{a^2 + (1 - a^2)e^{2t}}}.$$

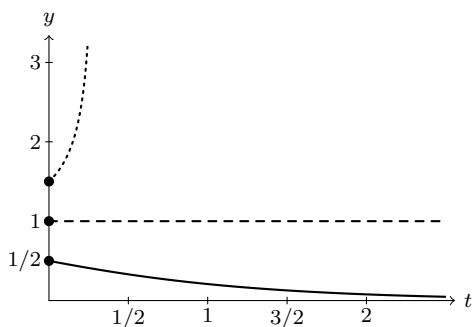
The denominator is the positive square root on the interval through 0. For $a = 0$, the unique solution is $y \equiv 0$; the smooth right-hand side $y^3 - y$ ensures local uniqueness.

Step 2: Classify the thresholds. If $|a| < 1$, the denominator is positive for every $t \geq 0$ and $y \rightarrow 0$. This includes $a = 0$. If $a = \pm 1$, the formula is constant: $y \equiv \pm 1$. If $|a| > 1$, the denominator first vanishes at

$$T = \frac{1}{2} \ln \frac{a^2}{a^2 - 1}.$$

The maximal forward lifetime is $0 \leq t < T$. As $t \uparrow T$, $y \rightarrow +\infty$ for $a > 1$ and $y \rightarrow -\infty$ for $a < -1$.

Step 3: Verify and interpret. For nonzero branches, $v' = -2y^{-3}y' = 2v - 2$ implies $y' = -y + y^3$, and the formula gives $y(0) = a$. The three constants 0, 1, -1 also check directly. At $a = \pm 1$ the exponential term in v is exactly zero; these solutions neither decay nor blow up. A threshold case must be substituted into the equation, not inferred from neighboring values.



Solid: $a = 1/2$. Dashed: $a = 1$. Dotted: $a = 3/2$. Negative solutions are reflections.

Question 6

On $x > 0$, two positive solutions of one unknown Bernoulli equation

$$y' + p(x)y = q(x)y^2$$

are known exactly:

$$y_1(x) = \frac{1}{1+x}, \quad y_2(x) = \frac{1}{1+2x}.$$

Assume p and q are continuous there.

Tasks

1. Transform both solutions using $v = 1/y$ and recover $p(x)$ and $q(x)$ uniquely.
2. Verify both supplied solutions directly in the recovered nonlinear equation.
3. Solve that equation with the different condition $y(1) = 2$, and find its maximal interval within $x > 0$.
4. Explain why subtracting the original nonlinear solutions is less useful here than subtracting their transformed equations, and identify the solution lost by taking reciprocals.

Question 6 – Solution

Strategy. The reciprocal transformation turns coefficient recovery into two linear identities; their difference eliminates the forcing term.

Step 1: Recover the coefficients. Each transformed solution satisfies $v' - pv = -q$. Here $v_1 = 1 + x$ and $v_2 = 1 + 2x$, so

$$1 - p(1 + x) = -q, \quad 2 - p(1 + 2x) = -q.$$

Subtracting gives $1 - px = 0$. Since $x > 0$, this fixes $p = 1/x$ uniquely. Substituting back gives $q = 1/x$. Thus

$$\boxed{y' + \frac{y}{x} = \frac{y^2}{x}}.$$

Step 2: Verify the given data. For $y = 1/(1 + kx)$, with $k = 1$ or 2 ,

$$y' + \frac{y}{x} = -\frac{k}{(1 + kx)^2} + \frac{1}{x(1 + kx)} = \frac{1}{x(1 + kx)^2} = \frac{y^2}{x}.$$

Both functions are positive and defined throughout $x > 0$.

Step 3: Construct the new solution. The transformed equation is $v' - v/x = -1/x$. Multiplying by $1/x$ gives $(v/x)' = -1/x^2$, hence $v = 1 + Cx$. Since $y(1) = 2$, $v(1) = 1/2$, so $C = -1/2$. Therefore

$$\boxed{y = \frac{1}{1 - x/2}, \quad I = (0, 2)}.$$

This is the maximal interval within the specified domain through 1: the solution has a pole at 2, and the coefficients are undefined at 0.

Step 4: Explain the advantage and restore zero. Subtracting the transformed equations gives the homogeneous linear relation $\Delta v' - p\Delta v = 0$. Subtracting the original equations leaves $q(y_2^2 - y_1^2)$ and does not give that linear elimination. The function $\boxed{y \equiv 0}$ satisfies the recovered equation but is excluded by the reciprocal transformation.

Question 7

Suppose y_1, y_2 are positive solutions on an interval of

$$y' + p(x)y = q(x)y^3.$$

Let $0 < \alpha < 1$.

Tasks

1. Prove that $v = \alpha y_1^{-2} + (1 - \alpha)y_2^{-2}$ solves the transformed linear equation and gives a positive solution $y = v^{-1/2}$.
2. Show that this new solution lies between y_1 and y_2 pointwise.
3. Apply the construction with $\alpha = 1/4$ to the equation $y' + xy = xy^3$ and the solutions $y_1 = 1$, $y_2 = (1 + e^{x^2})^{-1/2}$. Verify the supplied solutions through their transforms.
4. Prove that the ordinary arithmetic mean $m = (y_1 + y_2)/2$ of these two supplied solutions fails the nonlinear equation at $x = 1$.

Question 7 – Solution

Strategy. Affine combinations preserve the common forcing in the transformed linear equation; they do not generally preserve the original cubic equation.

Step 1: Combine the transformed solutions. Set $v_i = y_i^{-2}$. Both satisfy $v'_i - 2pv_i = -2q$. Multiplying by α and $1 - \alpha$ and adding yields $v' - 2pv = -2q$. Positivity of v_1, v_2 implies $v > 0$, so $y = v^{-1/2}$ is well defined. Differentiating this inverse recovers $y' + py = qy^3$.

Step 2: Prove the pointwise bound. At each point, v lies between v_1 and v_2 . The function $u \mapsto u^{-1/2}$ is decreasing on $u > 0$, so reversing their order still puts y between y_1 and y_2 . Equality occurs when the supplied values agree; otherwise the bounds are strict.

Step 3: Apply the construction. For the given equation, the transformed equation is $v' - 2xv = -2x$. The supplied transforms are $v_1 = 1$ and $v_2 = 1 + e^{x^2}$; direct differentiation verifies both. Their weighted combination gives

$$v = 1 + \frac{3}{4}e^{x^2}, \quad \boxed{y = \frac{1}{\sqrt{1 + \frac{3}{4}e^{x^2}}}} \quad (x \in \mathbb{R}).$$

It is positive and lies strictly between the two supplied solutions.

Step 4: Reject ordinary superposition. Because the original functions solve the equation,

$$m' + xm = \frac{x}{2}(y_1^3 + y_2^3).$$

At $x = 1$, put $a = y_1(1)$ and $b = y_2(1)$. They are distinct and positive. The residual is

$$m' + m - m^3 = \frac{a^3 + b^3}{2} - \frac{(a + b)^3}{8} = \frac{3}{8}(a + b)(a - b)^2 > 0.$$

Thus the arithmetic mean fails the nonlinear equation. The valid averaging occurs after the Bernoulli transformation.

Question 8

Consider

$$y' + xy = e^{x^2} y^2, \quad y(0) = a, \quad a \in \mathbb{R}.$$

Define $J(x) = \int_0^x e^{t^2/2} dt$; an elementary antiderivative is not required.

Tasks

1. Solve the IVP for $a \neq 0$ by the Bernoulli substitution, expressing the answer in terms of J . Treat $a = 0$ separately.
2. Prove that J is strictly increasing onto $\mathbb{R}$ and that the equation $J(T) = 1/a$ has exactly one real solution.
3. Determine the maximal interval containing 0 for each sign of a , and the sign of the divergence at its finite endpoint.
4. Verify the formula using the fundamental theorem of calculus and explain why an exact definite-integral expression is a complete answer.

Question 8 – Solution

Strategy. Keep the forcing integral exact, then use its monotonicity to determine the entire admissible denominator interval.

Step 1: Linearize and integrate. For $a \neq 0$, set $v = 1/y$. Then $v' - xv = -e^{x^2}$, with integrating factor $e^{-x^2/2}$. Thus

$$(e^{-x^2/2}v)' = -e^{x^2/2}, \quad v = e^{x^2/2} \left(\frac{1}{a} - J(x) \right).$$

Inverting gives

$$y(x) = \frac{ae^{-x^2/2}}{1 - aJ(x)}.$$

For $a = 0$, $y \equiv 0$ is the unique solution, since the original right-hand side is smooth in y .

Step 2: Locate the unique obstruction. By the fundamental theorem, $J' = e^{x^2/2} > 0$. The integrand is even, so J is odd. Also $J(x) \geq x$ for $x \geq 0$, so its limits at the two infinities are $\pm\infty$. Therefore it is one-to-one and onto, and there is exactly one T with $J(T) = 1/a$.

Step 3: Select the interval and sign. If $a > 0$, then $T > 0$, and the maximal interval is $(-\infty, T)$. The denominator approaches zero from above as $x \uparrow T$, so $y \rightarrow +\infty$. If $a < 0$, then $T < 0$, and the maximal interval is (T, ∞) ; as $x \downarrow T$, the denominator approaches zero from above and $y \rightarrow -\infty$. A pole cannot be crossed by a finite differentiable continuation.

Step 4: Verify and interpret. With $D = 1 - aJ$, differentiation using $J' = e^{x^2/2}$ gives

$$y' = -xy + \frac{a^2}{D^2} = -xy + e^{x^2} y^2.$$

Also $J(0) = 0$ gives $y(0) = a$. The definite integral specifies an exact differentiable function and uniquely determines the endpoint; elementary notation is unnecessary for either the solution or its interval.

Question 9

Seek strictly positive, 2π -periodic solutions, defined for every real x , of

$$y' + y = (2 + \sin x)y^2.$$

Tasks

1. Use the reciprocal substitution to find the general transformed solution.
2. Prove that there is exactly one strictly positive 2π -periodic solution and find its initial value at $x = 0$.
3. Determine its minimum and maximum over one period.
4. Show that changing the initial value to $y(0) = 1/2$ causes a finite positive-time blow-up, even though the periodic solution remains bounded. Verify the original equation for the periodic solution.

Question 9 – Solution

Strategy. Periodicity removes the exponentially growing homogeneous term in the reciprocal variable; positivity must then be checked before inversion.

Step 1: Solve the linear equation. With $v = 1/y$, the equation becomes $v' - v = -(2 + \sin x)$. A particular solution and the general solution are

$$v_p = 2 + \frac{\sin x + \cos x}{2}, \quad v = v_p + Ce^x.$$

Indeed $v'_p - v_p = -2 - \sin x$.

Step 2: Select and bound the periodic solution. If positive y is 2π -periodic, then so is $v = 1/y$. The term Ce^x is periodic only if $C = 0$. Since

$$2 - \frac{\sqrt{2}}{2} \leq v_p \leq 2 + \frac{\sqrt{2}}{2},$$

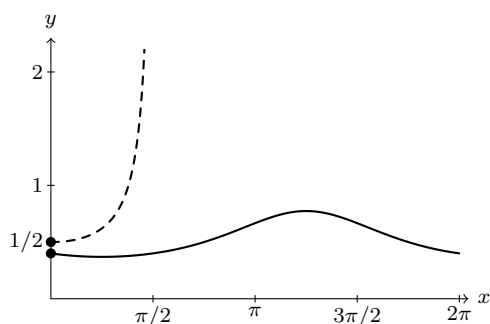
its reciprocal exists globally and is positive. Therefore

$$y_p = \frac{1}{2 + (\sin x + \cos x)/2}, \quad y_p(0) = \frac{2}{5}.$$

Its minimum is $1/(2 + \sqrt{2}/2)$ at $x = \pi/4$, and its maximum is $1/(2 - \sqrt{2}/2)$ at $x = 5\pi/4$, modulo 2π .

Step 3: Check the perturbed initial condition. For $y(0) = 1/2$, $v(0) = 2$, so $C = -1/2$. The function $v = v_p - e^x/2$ starts positive, but $v(2\pi) = 5/2 - e^{2\pi}/2 < 0$. It therefore has a first zero $T \in (0, 2\pi)$. At this zero, $v'(T) = -(2 + \sin T) < 0$. Hence $v \downarrow 0$ from the positive side and $y = 1/v \rightarrow +\infty$ as $x \uparrow T$.

Step 4: Verify the periodic branch. Since $v'_p = v_p - (2 + \sin x)$, differentiation gives $y'_p = -y_p + (2 + \sin x)y_p^2$, as required.



Solid: the positive periodic solution. Dashed: $y(0) = 1/2$, which blows up before a full cycle.

Question 10

Consider positive solutions, near $x = 0$, of the exponent-dependent IVP

$$y' + y = y^n, \quad y(0) = 2, \quad n \in \mathbb{R}.$$

For $y > 0$, interpret $y^n = \exp(n \ln y)$.

Tasks

1. Solve the cases $n = 0$ and $n = 1$ directly. Explain why they do not require the usual nonlinear Bernoulli reduction.
2. For $n \neq 1$, derive the solution formula and state the positivity condition needed to interpret its real power.
3. Check that the general formula agrees with the direct answer at $n = 0$ and verifies the initial condition for every admissible n .
4. For each fixed real x , prove that the formula tends to the $n = 1$ solution as $n \rightarrow 1$. Do not substitute $n = 1$ into an expression with exponent $1/(1 - n)$.

Question 10 – Solution

Strategy. Handle the exceptional equations directly, then analyze the apparent singularity of the general expression through its logarithm.

Step 1: Solve the elementary exceptional cases. If $n = 0$, the equation is $y' + y = 1$, giving $y = 1 + e^{-x}$. If $n = 1$, the two y terms cancel, giving $y = 2$. Both are positive on $\mathbb{R}$; the first is linear and the second has zero derivative.

Step 2: Derive the parameterized formula. For $n \neq 1$, set $h = 1 - n \neq 0$ and $v = y^h$. Then

$$v' + hv = h, \quad v(0) = 2^h,$$

so $v = 1 + (2^h - 1)e^{-hx}$. Therefore

$$y_n(x) = \left[1 + (2^{1-n} - 1)e^{-(1-n)x} \right]^{1/(1-n)}.$$

Use the connected interval through 0 on which the bracket is strictly positive. This is necessary because the original positive solution has $v = y^h > 0$; a negative base cannot be accepted by choosing a special rational exponent.

Step 3: Check the consistency conditions. At $n = 0$, the formula becomes $1 + e^{-x}$. At $x = 0$, its bracket is 2^{1-n} , and the positive-real power gives 2. Conversely, differentiating $v = y^h$ on the stated interval recovers the original equation, so the inversion is valid there.

Step 4: Take the limit without dividing by zero. For fixed x , let $B(h) = 1 + (e^{h \ln 2} - 1)e^{-hx}$. Since $B(0) = 1$, it is positive for all sufficiently small $|h|$. Now

$$\ln y_n = \frac{\ln B(h)}{h}, \quad B'(0) = \ln 2.$$

The derivative definition gives $\lim_{h \rightarrow 0} \ln B(h)/h = \ln 2$. Exponentiating yields $\lim_{n \rightarrow 1} y_n(x) = 2$, exactly the direct $n = 1$ solution. Joint continuity on the compact segment between 0 and the fixed x keeps the bracket positive there for small $|h|$, so these values belong to the IVP branch.