

Question 1

Consider the differential equation

$$(2xy + 1) dx + (x^2 + y^2) dy = 0, \quad y(1) = 1.$$

A potential for $M dx + N dy$ is a function F satisfying $F_x = M$ and $F_y = N$; its level curves $F = C$ give implicit solutions.

Tasks

1. Check exactness and construct a potential, retaining the unknown function that arises when integrating with respect to x .
2. Apply the initial condition and verify the resulting implicit solution by differentiation.
3. Find the slope and tangent line at $(1, 1)$.
4. Prove that the selected level determines a unique differentiable function $y(x)$ on all of $\mathbb{R}$, despite the vanishing of $x^2 + y^2$ at the origin.

Question 1 – Solution

Strategy. Recover a potential from both coefficients, then examine its derivative with respect to y on the selected level.

Step 1: Test and integrate. Here $M_y = 2x = N_x$, so the form is exact on $\mathbb{R}^2$. Integrating $F_x = M$ gives

$$F = x^2y + x + h(y), \quad F_y = x^2 + h'(y).$$

Matching N requires $h'(y) = y^2$, so we may choose $F = x^2y + x + y^3/3$. At $(1, 1)$, $F = 7/3$. Thus

$$\boxed{x^2y + x + \frac{y^3}{3} = \frac{7}{3}}.$$

Step 2: Verify and find the tangent. Differentiating this identity gives

$$2xy + 1 + (x^2 + y^2)y' = 0,$$

exactly the original equation for a graph. At $(1, 1)$ this yields $y' = -3/2$, hence $\boxed{y - 1 = -\frac{3}{2}(x - 1)}$.

Step 3: Establish a global graph. For fixed x , the function $u \mapsto x^2u + u^3/3$ is strictly increasing and ranges over all real numbers. This remains true at $x = 0$, where the cubic alone is strictly increasing. Therefore the equation

$$x^2u + \frac{u^3}{3} = \frac{7}{3} - x$$

has exactly one real root for every real x .

The derivative $F_y = x^2 + y^2$ can vanish only at $(0, 0)$, but that point has $F = 0$, not $7/3$. It is consequently nonzero everywhere on the selected level. The implicit function theorem gives a differentiable local graph at each point; the uniqueness of each root makes those graphs agree. They form one differentiable solution on all of $\mathbb{R}$. The isolated degeneracy on a different level does not obstruct this IVP.

Question 2

A student studies

$$(e^y + y^2) dx + (xe^y + 2xy + \cos y) dy = 0, \quad y(0) = 0.$$

After integrating the coefficient of dx , the student writes

$$F_*(x, y) = xe^y + xy^2$$

and claims that $F_* = 0$ solves the initial-value problem.

Tasks

1. Check exactness and identify precisely what is missing from the student's integration.
2. Construct a correct potential and select the correct level using the data.
3. Compute the local slope at $(0, 0)$ and explain why the student's proposed level cannot be the required solution graph.
4. Integrate the coefficient of dy first as an independent check. Explain why the two correct potentials can differ by a constant.

Question 2 – Solution

Strategy. A partial antiderivative permits an arbitrary function of the other variable; matching the second coefficient determines it.

Step 1: Locate the missing term. Both cross partials equal $e^y + 2y$, so the form is exact on $\mathbb{R}^2$. Integration with respect to x must give

$$F = xe^y + xy^2 + h(y).$$

Then $F_y = xe^y + 2xy + h'(y)$, so $h'(y) = \cos y$. We may take $h(y) = \sin y$; treating it as merely a numerical constant would omit a required derivative.

Step 2: Apply and test the initial condition. The correct potential is $F = xe^y + xy^2 + \sin y$, and $F(0, 0) = 0$. Thus

$$\boxed{x(e^y + y^2) + \sin y = 0}.$$

Its derivative along a graph is exactly the given equation. At the initial point, $M = 1$ and $N = 1$, so the local graph exists and $\boxed{y'(0) = -1}$.

In contrast, $F_* = 0$ says $x(e^y + y^2) = 0$. Since $e^y + y^2 > 0$, this is the vertical line $x = 0$, which cannot define $y(x)$ on an open interval. Also $(F_*)_y = N - \cos y$, so its differential is not the stated form.

Step 3: Check by reversing the integration order. Integrating N with respect to y gives

$$F = xe^y + xy^2 + \sin y + g(x).$$

Now $F_x = e^y + y^2 + g'(x) = M$ requires $g'(x) = 0$, leaving only an additive constant. Any two potentials for the same form have difference with both partial derivatives zero; on the connected plane that difference is constant. Changing this constant only relabels the same solution levels.

Question 3

Let $a, b \in \mathbb{R}$. Consider

$$(3x^2y^2 + 2bxy + e^y) dx + (2ax^3y + 5x^2 + xe^y + 4y^3) dy = 0.$$

Tasks

1. Determine every pair (a, b) for which the form is exact on $\mathbb{R}^2$. Explain why testing equality at a single point is insufficient.
2. For those parameters, construct a potential and verify both of its first partial derivatives.
3. Find the implicit solution through $(0, 1)$ and its tangent line there.
4. A student checks $M_y = N_x$ only at $(0, 0)$ and accepts every pair (a, b) . Show why that check cannot distinguish any of the parameters.

Question 3 – Solution

Strategy. Exactness requires an identity on a region, so compare the variable-dependent terms before constructing a potential.

Step 1: Determine the parameters. The cross partials are

$$M_y = 6x^2y + 2bx + e^y, \quad N_x = 6ax^2y + 10x + e^y.$$

Their difference is $6(1-a)x^2y + (2b-10)x$. For this to vanish for all (x, y) , first put $y = 0$ and $x = 1$, obtaining $b = 5$. Then put $x = y = 1$, obtaining $a = 1$. Conversely, those choices make the difference identically zero. Hence

$$\boxed{a = 1, \quad b = 5}.$$

Step 2: Construct and verify the potential. With these values, integrating M in x yields

$$F = x^3y^2 + 5x^2y + xe^y + h(y).$$

Matching F_y with N gives $h'(y) = 4y^3$. Thus choose

$$F = x^3y^2 + 5x^2y + xe^y + y^4.$$

Direct differentiation gives $F_x = 3x^2y^2 + 10xy + e^y = M$ and $F_y = 2x^3y + 5x^2 + xe^y + 4y^3 = N$.

Step 3: Apply the data and find the tangent. Since $F(0, 1) = 1$, the selected level is $\boxed{F(x, y) = 1}$. At that point, $N = 4 \neq 0$, so it determines a local solution graph. Its slope is $-M/N = -e/4$, giving

$$\boxed{y - 1 = -\frac{e}{4}x}.$$

Step 4: Explain the failed one-point test. At $(0, 0)$, both cross partials equal 1 for every a, b : all parameter-dependent terms vanish there. This agreement is necessary at that point but says nothing about agreement on a neighborhood. Exactness cannot be certified from that single sample.

Question 4

The coefficient $N(x, y)$ is unknown in

$$(2x + y) dx + N(x, y) dy = 0.$$

Assume N is continuously differentiable on $\mathbb{R}^2$, the form is exact there, and the calibration data are $N(0, y) = 2y$ for every real y .

Tasks

1. Recover N uniquely and construct a potential.
2. Select the solution through $(0, 1)$ and find an explicit formula for its graph near that point.
3. Determine the maximal open interval of this graph and its slope at $x = 0$.
4. Find the vertical-tangent points of the full selected level curve. Explain why the full curve is not a single differentiable function $y(x)$.

Question 4 – Solution

Strategy. Use exactness as an equation for the missing coefficient, then distinguish a complete level curve from its graph branches.

Step 1: Recover the form. Exactness gives $N_x = M_y = 1$, so $N = x + h(y)$. The data force $h(y) = 2y$, uniquely. A potential is

$$N = x + 2y, \quad F = x^2 + xy + y^2.$$

Its derivatives are $2x + y$ and $x + 2y$, as required.

Step 2: Select a branch. The initial point gives $F = 1$. Solving the quadratic in y and selecting $y(0) = 1$ yields

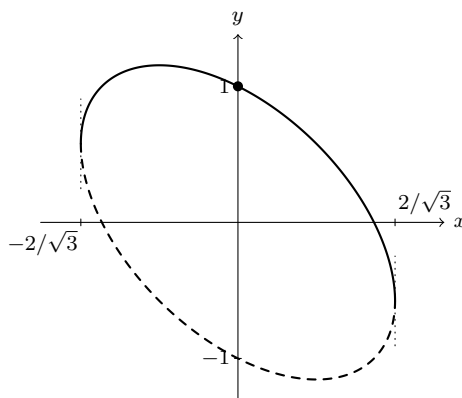
$$y = \frac{-x + \sqrt{4 - 3x^2}}{2}, \quad I = (-2/\sqrt{3}, 2/\sqrt{3}).$$

At $(0, 1)$, $y' = -(2x + y)/(x + 2y) = -1/2$.

Step 3: Interpret the endpoints and full curve. On this branch, $N = \sqrt{4 - 3x^2} > 0$. At the two endpoints, $N = 0$, while $M = 3x/2 \neq 0$. Thus a finite y' cannot satisfy $M + Ny' = 0$ there. The full ellipse has vertical tangents at

$$(2/\sqrt{3}, -1/\sqrt{3}), \quad (-2/\sqrt{3}, 1/\sqrt{3}).$$

It remains a smooth curve at these points, but cannot be continued through either as a differentiable graph over x . Most vertical lines through the ellipse meet both quadratic branches, so the complete ellipse is not one function $y(x)$.



Solid: IVP branch, excluding its vertical-tangent endpoints. Dashed: the other half.

Question 5

Consider the exact equation

$$-2x \, dx + 2y \, dy = 0, \quad y(0) = 0.$$

In this problem a solution means a real, continuously differentiable function $y(x)$ on an open interval containing 0, satisfying $-2x + 2yy' = 0$ at every point, including 0.

Tasks

1. Find a potential and the level selected by the initial condition.
2. Classify all solution graphs on such an interval. Justify why no additional sign-switching choices are differentiable at 0.
3. Check $y = x$, $y = -x$, $y = |x|$ and $y = -|x|$ against the stated definition.
4. Explain why neither division by y nor the usual nonzero- F_y implicit-function criterion settles the initial point.

Question 5 – Solution

Strategy. The selected level has crossing branches. Analyze differentiability at their intersection directly in the original equation.

Step 1: Find the level. Since $M_y = N_x = 0$, a potential is $F = y^2 - x^2$. Along any solution, $dF/dx = 0$, and the initial condition gives

$$\boxed{y^2 - x^2 = 0}.$$

Thus for each $x \neq 0$, y/x is either 1 or -1 .

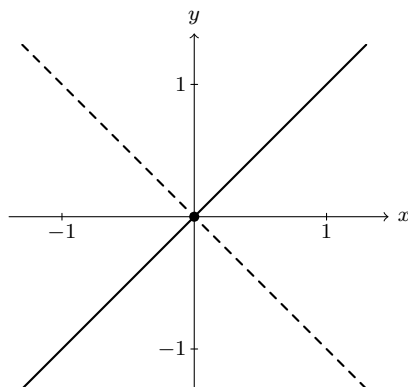
Step 2: Classify differentiable choices. Continuity makes y/x constant on each of the two connected pieces $x < 0$ and $x > 0$: a continuous function taking only the values ± 1 cannot switch between them. Write those constants as s_- and s_+ . At 0, the left and right difference quotients are s_- and s_+ . Differentiability requires them to agree. Therefore the only possibilities are

$$\boxed{y = x \quad \text{or} \quad y = -x}.$$

Both extend to all of $\mathbb{R}$.

Step 3: Verify and identify the failures. For either line, $yy' = x$ everywhere, including 0. The choices $|x|$ and $-|x|$ satisfy the level identity and the equation away from 0, but their one-sided derivatives disagree at 0. They fail the required definition.

Step 4: Explain the degenerate point. Dividing gives $y' = x/y$, which is undefined at the initial point and discards information about solutions of the original form there. Also $F_y = 2y = 0$ at $(0, 0)$, so the nonzero- F_y criterion does not apply. Its failure does not prove nonexistence: here two valid graph branches pass through the point.



Two differentiable solutions cross at $(0, 0)$: solid $y = x$, dashed $y = -x$.

Question 6

On the half-plane $x > 0$, consider

$$(2y + x) dx + x dy = 0, \quad y(1) = 2.$$

An integrating factor is a nonzero function that makes a differential form exact when multiplied into both coefficients.

Tasks

1. Show that the original form is not exact. Derive, using the product rule, the condition on an integrating factor $\mu(x)$ depending only on x .
2. Find every such nonzero factor on $x > 0$, choose one, and construct a potential for the multiplied form.
3. Solve and verify the initial-value problem on $x > 0$.
4. Now consider the original equation on the whole plane. Decide whether the selected solution can extend through $x = 0$, and identify which member of the implicit solution family does extend differentiably through that point.

Question 6 – Solution

Strategy. Derive the integrating-factor condition from exactness rather than importing a formula, and return to the original equation at a zero of the chosen factor.

Step 1: Derive the factor. Originally $M_y = 2$ while $N_x = 1$. For $\mu = \mu(x)$, exactness requires

$$(\mu M)_y = 2\mu = (\mu x)_x = x\mu' + \mu.$$

Thus $x\mu' = \mu$. On $x > 0$, its nonzero solutions are $\mu = Kx$, $K \neq 0$. Choose $\boxed{\mu = x}$, which is nonzero throughout the stated half-plane.

Step 2: Integrate the exact form. The multiplied coefficients are $2xy + x^2$ and x^2 . Integrating in y gives

$$F = x^2y + g(x), \quad F_x = 2xy + g'(x) = 2xy + x^2.$$

Hence $g = x^3/3$ is a choice, giving the family $\boxed{x^2y + x^3/3 = C}$.

Step 3: Apply and verify the data. At $(1, 2)$, $C = 7/3$, so

$$\boxed{y = \frac{7}{3x^2} - \frac{x}{3}, \quad x > 0}.$$

Its derivative is $-14/(3x^3) - 1/3$, and substitution gives

$$2y + x + xy' = \frac{14}{3x^2} - \frac{2x}{3} + x - \frac{14}{3x^2} - \frac{x}{3} = 0.$$

The initial value is also 2.

Step 4: Check extension in the original equation. The selected solution tends to $+\infty$ as $x \downarrow 0$, so it cannot extend continuously through 0. In the general family, $y = C/x^2 - x/3$, a finite extension requires $C = 0$. This member extends as $\boxed{y = -x/3}$ on all of $\mathbb{R}$ and satisfies the original equation even at 0, where the equation requires $2y(0) = 0$. The multiplied form alone gives $0 = 0$ there because its factor vanishes; that weakened condition must not replace the original one.

Question 7

Compare the following forms on $\mathbb{R}^2$:

(A) $x \, dx + y \, dy = 0$,

(B) $e^x x \, dx + e^x y \, dy = 0$,

(C) $(1 + x^2 + y^2)x \, dx + (1 + x^2 + y^2)y \, dy = 0$.

Tasks

1. Determine which forms are exact and give a potential for each exact form.
2. Prove that all three nevertheless have the same differentiable solution graphs.
3. Find their common solution graph with $y(0) = 2$ and its maximal open interval containing 0.
4. Let $dF = M \, dx + N \, dy$. Explain why multiplication by a continuous function $h(F)$ produces another exact form, and state an additional condition that guarantees equivalence of the graph equations.

Question 7 – Solution

Strategy. Separate two different questions: whether the displayed coefficients are partial derivatives of a potential, and whether multiplying the equation changes its solutions.

Step 1: Test each form. For (A), $M_y = N_x = 0$, with potential

$$F = \frac{x^2 + y^2}{2}.$$

For (B), $M_y = 0$ and $N_x = e^x y$, which are not equal on the plane, so (B) is not exact there. For (C), both cross partials are $2xy$, and a potential is

$$G = \frac{x^2 + y^2}{2} + \frac{(x^2 + y^2)^2}{4}.$$

Differentiating G gives the two coefficients in (C).

Step 2: Establish equivalence. The equations for a graph are respectively

$$x + yy' = 0, \quad e^x(x + yy') = 0, \quad (1 + x^2 + y^2)(x + yy') = 0.$$

Both multiplying factors are strictly positive at every real point. Each equation therefore holds exactly when the first does. Exactness of a particular form is not required for it to share the same solutions as an exact form.

Step 3: Select the common IVP solution. The first potential gives $x^2 + y^2 = 4$. Continuity from $y(0) = 2$ selects

$$\boxed{y = \sqrt{4 - x^2}, \quad I = (-2, 2)}.$$

Here $y' = -x/\sqrt{4 - x^2}$ verifies all three equations. At either endpoint, $y = 0$ but $x \neq 0$, so no finite derivative can satisfy $x + yy' = 0$; a differentiable graph cannot extend through it.

Step 4: Generalize using the chain rule. Choose an antiderivative H of the continuous function h . Then

$$d(H(F)) = H'(F) dF = h(F)(M dx + N dy).$$

This gives a potential for the new form. If $h(F)$ is nonzero throughout the region considered, division is valid there and the graph equations are equivalent. Allowing a zero factor can introduce extra solutions, so equivalence cannot be concluded from exactness alone.

Question 8

On the domain $xy \neq 0$, consider

$$\left(\frac{1}{x} + y\right) dx + \left(\frac{1}{y} + x\right) dy = 0, \quad y(1) = 2.$$

Tasks

1. Find a potential valid on this domain, using absolute values wherever required by logarithmic integration.
2. Use the initial condition to identify the correct level. On the positive quadrant, prove that the implicit relation reduces to an explicit solution.
3. Find its maximal interval containing 1. Explain why the branch of the same algebraic hyperbola in the negative quadrant is not an extension of this IVP.
4. Analyze the different initial condition $y(1) = -1$. Determine its selected level and solution on $x > 0$, even though both coefficients vanish along that solution.

Question 8 – Solution

Strategy. Combine the logarithms using the product $u = xy$, but keep the connected domain and the sign of u in view.

Step 1: Construct the potential. Both cross partials are 1. Integration gives

$$F = \ln |x| + \ln |y| + xy = \ln |xy| + xy.$$

Indeed $F_x = 1/x + y$ and $F_y = 1/y + x$ wherever $xy \neq 0$.

Step 2: Select the positive branch. The data $(1, 2)$ give $F = \ln 2 + 2$. For $u > 0$, the function $\phi(u) = \ln u + u$ is strictly increasing, since $\phi'(u) = 1/u + 1 > 0$. Thus $\phi(u) = \phi(2)$ forces $u = 2$. Consequently

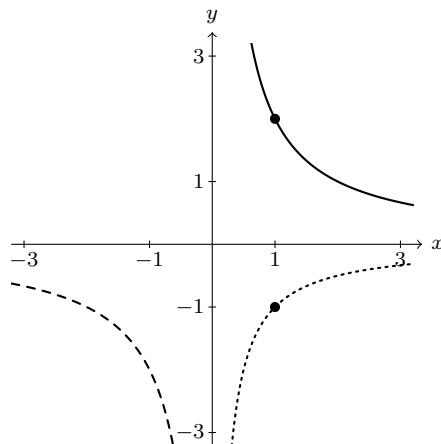
$$y = 2/x, \quad I = (0, \infty).$$

Substitution gives $M = 3/x$, $N = 3x/2$ and $y' = -2/x^2$, so $M + Ny' = 0$. The branch cannot cross $x = 0$: the equation is undefined there and $y \rightarrow +\infty$ from the right. The negative-quadrant branch $y = 2/x$, $x < 0$, is disconnected from the initial point.

Step 3: Analyze the degenerate negative level. For $(1, -1)$, $F = -1$. On $u < 0$, $\phi(u) = \ln(-u) + u$ has derivative $1/u + 1$, positive for $u < -1$ and negative for $-1 < u < 0$. Its unique maximum is $\phi(-1) = -1$. Therefore this level forces $xy = -1$, giving

$$y = -1/x \quad (x > 0).$$

Along this graph both M and N are zero, so the original equation is satisfied directly. Every solution through the point must conserve F and remain in the same quadrant, which proves this branch is unique there despite the vanishing coefficients.



Solid: $y(1) = 2$. Dashed: its disconnected hyperbola branch. Dotted: $y(1) = -1$.

Question 9

Consider

$$e^x dx + (1 + y^2) dy = 0, \quad y(0) = 0.$$

An approximate calculation claims that the selected solution passes through $(\ln 2, -1)$.

Tasks

1. Find the exact implicit solution and prove that it defines a unique differentiable graph for all real x .
2. Test the claimed point using the potential, without explicitly solving a cubic.
3. Let y_* be the true value at $x = \ln 2$. Prove $-1 < y_* < 0$ and decide whether the claimed value is too high or too low.
4. Use the mean value theorem to prove the quantitative error bound

$$\frac{1}{6} \leq |y_* - (-1)| \leq \frac{1}{3}.$$

Explain why a potential residual can provide more information than simply declaring an approximate point incorrect.

Question 9 – Solution

Strategy. Use the conserved potential to test the point, then convert its residual into an error bound using a derivative estimate.

Step 1: Find the exact relation. The cross partials are both zero. A potential is $F = e^x + y + y^3/3$, and $F(0, 0) = 1$, so

$$e^x + y + \frac{y^3}{3} = 1.$$

The function $H(u) = u + u^3/3$ is strictly increasing, onto $\mathbb{R}$, and has $H'(u) = 1 + u^2 > 0$. Thus $H(y) = 1 - e^x$ gives exactly one differentiable value for every x . Differentiation recovers the original graph equation.

Step 2: Compute the residual and locate the true value. At the claimed point, $F = 2 - 1 - 1/3 = 2/3$, so the potential residual $F - 1$ is $-1/3$. The point is not on the selected level. At $x = \ln 2$, the true value satisfies $H(y_*) = -1$. Since $H(-1) = -4/3 < -1 < H(0) = 0$, strict monotonicity gives $-1 < y_* < 0$. The claimed value -1 is too low.

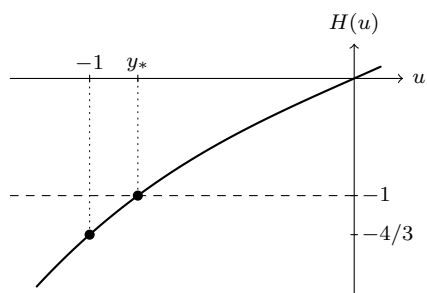
Step 3: Bound the actual error. The mean value theorem gives, for some $\xi \in (-1, y_*)$,

$$\frac{1}{3} = H(y_*) - H(-1) = H'(\xi)(y_* + 1).$$

On $[-1, 0]$, $1 \leq H' \leq 2$, so

$$\frac{1}{6} \leq y_* + 1 = |y_* - (-1)| \leq \frac{1}{3}.$$

The residual, together with monotonicity and derivative bounds, certifies both the direction and the size range of the error; no cubic formula is needed.



The true value y_* lies to the right of -1 because H is strictly increasing.

Question 10

For the exact form

$$\omega = (2xy + e^x) dx + (x^2 + 2y) dy,$$

compare two routes from $A = (0, 0)$ to $B = (1, 1)$. Route 1 goes horizontally to $(1, 0)$ and then vertically to B ; Route 2 goes vertically to $(0, 1)$ and then horizontally to B .

For an axis-aligned segment, define its accumulated value as the integral of the coefficient of dx along a horizontal segment, or of the coefficient of dy along a vertical segment, with the stated orientation.

Tasks

1. Construct a potential F normalized by $F(0, 0) = 0$.
2. Compute both route totals directly as sums of ordinary single-variable integrals.
3. Explain their equality using the chain rule and find the total around the closed route that follows Route 1 forward and Route 2 backward.
4. For the differential equation $\omega = 0$, decide whether one solution curve could pass through both A and B . Distinguish this question from the equality of the two route totals.

Question 10 – Solution

Strategy. A potential measures accumulated change along a route; a solution of the equation $\omega = 0$ must instead keep that potential constant.

Step 1: Construct a normalized potential. The cross partials are $2x$. Integrating M gives $F = x^2y + e^x + h(y)$; matching N requires $h' = 2y$. Normalization yields

$$\boxed{F = x^2y + e^x + y^2 - 1}, \quad F(B) = e + 1.$$

Step 2: Compute the routes directly. On Route 1, the horizontal and vertical contributions are

$$\int_0^1 e^x dx = e - 1, \quad \int_0^1 (1 + 2y) dy = 2,$$

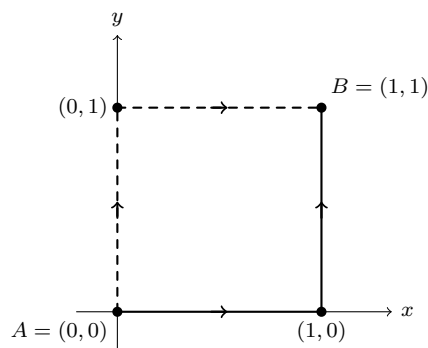
so its total is $e + 1$. On Route 2 they are

$$\int_0^1 2y dy = 1, \quad \int_0^1 (2x + e^x) dx = e,$$

again totaling $\boxed{e + 1}$.

Step 3: Explain equality and the closed route. Along a differentiable parametrized segment, $dF/dt = F_x x'(t) + F_y y'(t) = Mx' + Ny'$. Integration gives the endpoint difference of F ; these differences telescope across joined segments. Both routes therefore give $F(B) - F(A) = e + 1$. Reversing a route negates its integrals, so the closed-route total is $\boxed{0}$.

Step 4: Distinguish routes from solutions. A solution curve of $\omega = 0$ has $dF/dt = 0$ and stays on one level. Since $F(A) = 0$ and $F(B) = e + 1 \neq 0$, no such curve can pass through both points. Equal route totals express endpoint dependence; they do not make either route a solution of the differential equation.



Solid: Route 1. Dashed: Route 2. Arrows indicate travel from A toward B .