

Question 1

Consider the initial-value problem

$$y' = 2x(1 + y^2), \quad y(0) = 1.$$

Tasks

1. Solve by separation, explaining how the initial condition selects the integration constant.
2. Find the maximal open interval containing 0 on which the solution is finite and differentiable.
3. Determine its monotonicity, minimum, and behavior at both interval endpoints.
4. A student says that the periodicity of tangent extends this same solution through each vertical asymptote. Explain the error and verify the valid solution directly.

Question 1 – Solution

Strategy. Separate using the everywhere-positive factor $1 + y^2$, then respect the range of the inverse tangent.

Step 1: Integrate and select the branch.

$$\frac{dy}{1+y^2} = 2x \, dx, \quad \arctan y = x^2 + C, \quad C = \frac{\pi}{4}.$$

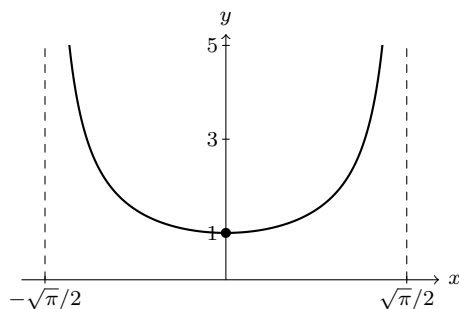
Thus $y(x) = \tan(x^2 + \pi/4)$ near 0. No constant solution was lost, since $1 + y^2$ never vanishes for real y .

Step 2: Identify the interval and shape. For the branch through 0, we need $-\pi/2 < x^2 + \pi/4 < \pi/2$. Hence

$$I = (-\sqrt{\pi}/2, \sqrt{\pi}/2).$$

The derivative has the sign of x : the solution decreases to its minimum $y(0) = 1$ and then increases. At either endpoint approached from inside I , the angle tends to $\pi/2$ from below, so $y \rightarrow +\infty$.

Step 3: Verify and explain the obstruction. Differentiation gives $y' = 2x \sec^2(x^2 + \pi/4) = 2x(1 + y^2)$, and $y(0) = 1$. A real differentiable extension through an endpoint would have a finite, continuous value there, contradicting blow-up. Other tangent branches lie on separate intervals; periodicity does not connect them across a pole.



The branch through $(0, 1)$ has a pole at each dashed boundary.

Question 2

For the autonomous equation

$$y' = y(3 - y),$$

consider the initial condition $y(0) = 4$.

Tasks

1. Find every constant solution before dividing by any expression involving y .
2. Obtain the nonconstant solution family by separation and solve the given initial-value problem.
3. Find its maximal interval containing 0, its direction of motion, and its limiting values at the ends of that interval.
4. Find the exact positive time when $y = 7/2$. Explain why this solution never reaches or crosses $y = 3$ at a finite time.

Question 2 – Solution

Strategy. Preserve the two equilibrium solutions, then use partial fractions on a nonconstant branch.

Step 1: Separate without losing constants. The constant solutions are $y \equiv 0, y \equiv 3$. Away from those values,

$$\frac{1}{y(3-y)} = \frac{1}{3y} + \frac{1}{3(3-y)}, \quad \frac{1}{3} \ln \left| \frac{y}{3-y} \right| = x + C.$$

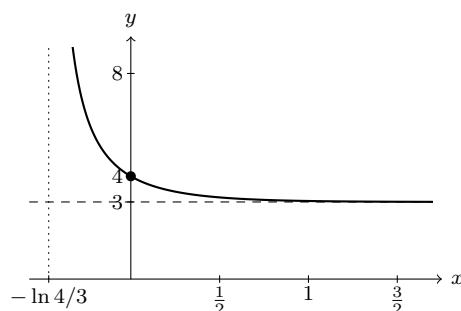
The nonconstant family can be written $y = 3/(1 + Ae^{-3x})$, $A \neq 0$, on intervals where the denominator is nonzero. The value $A = 0$ restores $y \equiv 3$; $y \equiv 0$ must still be listed separately.

Step 2: Apply the data and determine the domain. From $4 = 3/(1 + A)$, $A = -1/4$, so

$$y(x) = \frac{3}{1 - \frac{1}{4}e^{-3x}}, \quad I = (-\ln 4/3, \infty).$$

Here $y > 3$, so $y' < 0$. The limits are $+\infty$ at the left endpoint and 3 as $x \rightarrow \infty$. Writing $D = 1 - \frac{1}{4}e^{-3x}$ gives $y' = -9e^{-3x}/(4D^2) = y(3-y)$, verifying the equation.

Step 3: Compute the passage time. The equation $7/2 = 3/(1 - \frac{1}{4}e^{-3x})$ gives $x = \frac{1}{3} \ln(7/4)$. At any finite $x \in I$, the exponential is positive, so $y - 3 > 0$: the limiting level is never attained or crossed.



The decreasing solution stays above the dashed equilibrium $y = 3$.

Question 3

Consider an equation whose right-hand side is undefined on the line $y = 2$:

$$y' = -\frac{e^x}{y-2}, \quad y(0) = 1.$$

Tasks

1. Separate the equation and obtain an implicit relation.
2. Select the correct explicit branch and determine its maximal interval containing 0.
3. Find the limiting value of y and the behavior of y' at the finite endpoint. Decide whether that endpoint can be included.
4. Another student takes the opposite square-root sign, then proposes switching signs at the endpoint. Test both claims against the equation and initial condition.

Question 3 – Solution

Strategy. Integration gives a squared relation; the initial condition and the excluded line determine which part is a solution.

Step 1: Integrate and choose the sign.

$$(y - 2) dy = -e^x dx, \quad \frac{1}{2}(y - 2)^2 = -e^x + C.$$

At $(0, 1)$, $C = 3/2$, so $(y - 2)^2 = 3 - 2e^x$. Since $y(0) - 2 = -1$, continuity selects

$$y = 2 - \sqrt{3 - 2e^x}, \quad I = (-\infty, \ln(3/2)).$$

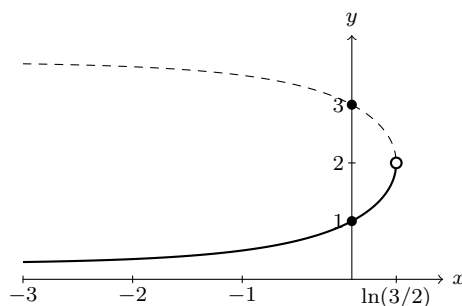
The radicand must be strictly positive: zero would give the forbidden value $y = 2$.

Step 2: Check behavior and verify. Throughout I ,

$$y' = \frac{e^x}{\sqrt{3 - 2e^x}} = -\frac{e^x}{y - 2} > 0.$$

As $x \rightarrow -\infty$, $y \rightarrow 2 - \sqrt{3}$. As $x \uparrow \ln(3/2)$, $y \rightarrow 2$ but $y' \rightarrow +\infty$. A finite limit of y alone does not permit extension: the equation is undefined at the endpoint and a finite derivative is impossible there.

Step 3: Audit the other branch. The branch $2 + \sqrt{3 - 2e^x}$ satisfies the differential equation on I , but has $y(0) = 3$, not 1. Switching signs would still pass through $y = 2$, and for larger x the radicand is negative. Neither proposal extends the given real solution.



Solid: $y(0) = 1$. Dashed: $y(0) = 3$. The common limiting point is excluded.

Question 4

An explicit elementary formula is not required in this problem:

$$y' = \frac{x}{1 + e^y}, \quad y(0) = 0.$$

Tasks

1. Use separation to find an implicit equation for the solution.
2. Prove that it determines exactly one real value of y for every real x , and that the resulting function is differentiable.
3. Determine its symmetry, monotonicity, and minimum. Prove the bound $0 \leq y(x) \leq x^2/4$ and identify every equality case.
4. Verify the initial condition and differential equation, and explain why failure to isolate y in elementary functions is not a failure to solve the problem.

Question 4 – Solution

Strategy. Treat the left side of the integrated equation as an invertible function rather than forcing an elementary inverse.

Step 1: Integrate. Multiplication and integration give

$$(1 + e^y) dy = x dx, \quad \boxed{y + e^y = 1 + \frac{x^2}{2}}.$$

The constant is 1 because $y(0) = 0$.

Step 2: Prove global existence and uniqueness of the implicit value. Let $F(u) = u + e^u$. It is continuous and strictly increasing because $F'(u) = 1 + e^u > 0$. Also $F(u) \rightarrow -\infty$ as $u \rightarrow -\infty$ and $F(u) \rightarrow \infty$ as $u \rightarrow \infty$. It therefore has an inverse on all of $\mathbb{R}$. Since its derivative never vanishes, the inverse is differentiable. Thus

$$y(x) = F^{-1}(1 + x^2/2)$$

defines a unique differentiable solution for every real x . Any solution through $(0, 0)$ must satisfy the same integrated identity, establishing uniqueness for the initial-value problem.

Step 3: Extract shape and an exact bound. Uniqueness of the implicit value and the dependence on x^2 show $y(-x) = y(x)$. Since $F(0) = 1$, $y \geq 0$, with equality only at $x = 0$. The differential equation makes y decreasing for $x < 0$ and increasing for $x > 0$, so its unique minimum is 0.

Using $e^y \geq 1 + y$, with equality only at $y = 0$, gives

$$1 + \frac{x^2}{2} = y + e^y \geq 1 + 2y, \quad \boxed{0 \leq y \leq \frac{x^2}{4}}.$$

Equality in either bound occurs only at $x = 0$.

Step 4: Verify and interpret. Implicit differentiation yields $(1 + e^y)y' = x$, exactly the given equation. At $x = 0$, strict monotonicity of F forces $y = 0$. The implicit formula uniquely specifies every solution value, its full domain, and its derivative; an elementary inverse is unnecessary.

Question 5

Consider these two equations for real x and y :

$$(A) \quad y' = x + y, \quad (B) \quad y' = xy + x + y + 1.$$

Here “directly separable” means that the right-hand side can be written $g(x)h(y)$ on the region in question, without a change of variables.

Tasks

1. Prove that any product $f(x, y) = g(x)h(y)$ satisfies

$$f(a, c)f(b, d) = f(a, d)f(b, c).$$

2. Use $a = c = 1$ and $b = d = 2$ to show that (A) is not directly separable on the rectangle $[1, 2] \times [1, 2]$.
3. Factor the right-hand side of (B), then find all its real solutions on intervals and list any constant solution separately.
4. Solve (B) with $y(0) = -2$, verify it, and explain why division by $x + 1$ is unnecessary even at $x = -1$.

Question 5 – Solution

Strategy. Test whether variables actually factor before performing separation; a sum of terms can still conceal a product.

Step 1: Establish the necessary product test. If $f(x, y) = g(x)h(y)$, then both products in the stated identity equal $g(a)g(b)h(c)h(d)$. This argument does not divide by any factor and remains valid if some factors vanish.

Step 2: Reject direct separation for (A). For $f(x, y) = x + y$, the proposed rectangle gives

$$f(1, 1)f(2, 2) = 2 \cdot 4 = 8, \quad f(1, 2)f(2, 1) = 3 \cdot 3 = 9.$$

The necessary identity fails, so there is no product representation on this rectangle. This conclusion concerns direct separation on that region; it makes no claim about other methods of solving (A).

Step 3: Solve the factored equation (B). Here $xy + x + y + 1 = (x + 1)(y + 1)$. First, $y \equiv -1$ is a constant solution. For $y \neq -1$,

$$\frac{dy}{y + 1} = (x + 1) dx, \quad \ln |y + 1| = \frac{x^2}{2} + x + C.$$

Absorbing the fixed sign into a nonzero constant gives

$$\boxed{y = -1 + Ae^{x^2/2+x}}.$$

Allowing $A = 0$ includes the constant solution as well. These are all solutions: for any differentiable solution, the product rule applied to $(y + 1)e^{-x^2/2-x}$ gives derivative zero, even at $y = -1$.

Step 4: Apply and check the data. The initial condition requires $A = -1$, so $\boxed{y = -1 - e^{x^2/2+x}}$ on all of $\mathbb{R}$. Its derivative is $-(x + 1)e^{x^2/2+x} = (x + 1)(y + 1)$, and $y(0) = -2$. Separation integrates the factor $x + 1$; it never requires its reciprocal. The point $x = -1$ is an ordinary point with $y' = 0$.

Question 6

A nonnegative quantity satisfies

$$y' = -3y^{2/3}, \quad y(0) = 8, \quad t \geq 0,$$

where the prime means d/dt . For real y , define $y^{2/3} = (\sqrt[3]{y})^2$.

Tasks

1. Solve while $y > 0$ and find the first time the quantity reaches zero.
2. Find and justify the unique continuation for all $t \geq 0$ if the solution must remain nonnegative.
3. If negative values are allowed, classify all continuously differentiable continuations after the first zero, including solutions that wait at zero before becoming negative.
4. Check the equation at every joining time and identify exactly what is lost by dividing by $y^{2/3}$.

Question 6 – Solution

Strategy. Separate only where $y \neq 0$, then use the sign of y' and check any joins at zero directly.

Step 1: Reach zero. While $y > 0$, integration gives $3y^{1/3} = -3t + C$. The data yield $y = (2 - t)^3$ for $0 \leq t < 2$, so the first zero is $t = 2$.

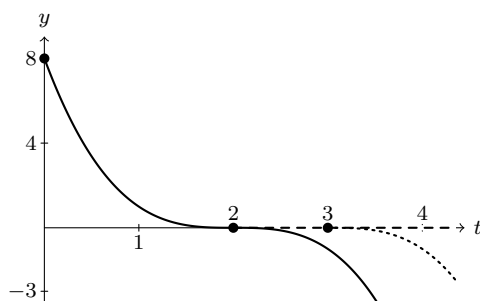
Step 2: Enforce nonnegativity. The equation always gives $y' \leq 0$. Once a nonnegative solution reaches zero it cannot increase or decrease without violating that constraint. Thus the unique nonnegative continuation is $(2 - t)^3$ up to 2, followed by 0 forever.

Step 3: Classify real continuations. For any finite $b \geq 2$, a real continuation is

$$y_b(t) = \begin{cases} (2 - t)^3, & 0 \leq t \leq 2, \\ 0, & 2 \leq t \leq b, \\ -(t - b)^3, & t \geq b. \end{cases}$$

Also allow $b = \infty$, meaning the solution stays zero forever. These exhaust the possibilities: monotonicity makes the zero set after 2 an interval; once negative, separation gives $\sqrt[3]{y} = b - t$, where continuity fixes the constant at the departure time b .

Step 4: Verify the joins and the lost solutions. Each cubic piece has $y' = -3(\sqrt[3]{y})^2$. At $t = 2$ and any finite b , both one-sided derivatives are 0, matching the equation at $y = 0$. The pieces are therefore C^1 . Division excludes zero itself, so it misses the constant zero solution and all zero waiting intervals; using one integration constant through zero is unjustified.



After $t = 2$: solid $b = 2$, dotted $b = 3$, dashed $b = \infty$ (zero forever).

Question 7

A positive concentration obeys a power-law decay model

$$\frac{dy}{dt} = -ky^p, \quad k > 0, \quad p \in \mathbb{R}.$$

Use fixed units for time and concentration. Exact observations are

$$y(0) = 16, \quad y(1) = 8, \quad y(2) = 4.$$

Assume one pair of constants (k, p) describes the entire observation interval.

Tasks

1. Express each of the two observed one-unit decay times as a separated definite integral, without assuming $p = 1$.
2. Prove that these observations determine p uniquely. Treat the case $p = 1$ without dividing by $1 - p$.
3. Find k and the resulting solution, and predict when the concentration first reaches 1.
4. Explain why the first two observations alone would not determine p .

Question 7 – Solution

Strategy. Compare passage-time integrals on two concentration intervals related by scaling; the unknown constant k then cancels.

Step 1: Write the two passage times. Since $y > 0$ and $k > 0$, the concentration decreases strictly. Separation gives

$$1 = \frac{1}{k} \int_8^{16} s^{-p} ds, \quad 1 = \frac{1}{k} \int_4^8 s^{-p} ds.$$

These integrals are positive and finite for every real p .

Step 2: Recover the exponent. In the first integral put $s = 2u$:

$$\int_8^{16} s^{-p} ds = 2^{1-p} \int_4^8 u^{-p} du.$$

The observed equal passage times force $2^{1-p} = 1$, hence $\boxed{p = 1}$. Conversely, $p = 1$ makes the two integrals equal. No division by $1 - p$ was used, so the logarithmic case was retained throughout.

Step 3: Determine the rate and prediction. For $p = 1$,

$$k = \int_8^{16} \frac{ds}{s} = \ln 2, \quad \ln \frac{y}{16} = -t \ln 2.$$

Therefore $\boxed{y(t) = 16 2^{-t}}$. It gives exactly 16, 8, 4 at the measured times, and differentiating gives $y' = -(\ln 2)y$. It is positive for every real t . The value 1 is first reached at $\boxed{t = 4}$ because the solution is strictly decreasing.

Step 4: Identify the information supplied by the third observation. With only $y(0) = 16$ and $y(1) = 8$, each real p could be paired with

$$k(p) = \int_8^{16} s^{-p} ds > 0.$$

The corresponding separated solution decreases from 16 to 8 in one unit of time and stays positive during that interval. Thus infinitely many exponents fit the first two data; the second measured halving time identifies the exponent within this model family.

Question 8

Consider the parameterized initial-value problem

$$y' = (\sin x)y^2, \quad y(0) = a, \quad a > 0.$$

Tasks

1. Solve by separation and determine the maximal open interval containing 0 for every positive a .
2. Find the precise threshold separating global bounded solutions from finite-endpoint blow-up. Include the threshold case.
3. For the global solutions, find the minimum, maximum, and a period.
4. A student argues that $\int_0^{2\pi} \sin x \, dx = 0$ guarantees $y(2\pi) = a$ for every $a > 0$. Explain exactly when this reasoning is valid and why it can fail.

Question 8 – Solution

Strategy. Track the denominator of the separated formula throughout an interval, not merely at its endpoints.

Step 1: Integrate. Since $a > 0$, separation is valid near 0:

$$-\frac{1}{y} = -\cos x + C, \quad \boxed{y = \frac{a}{1 - a + a \cos x}}.$$

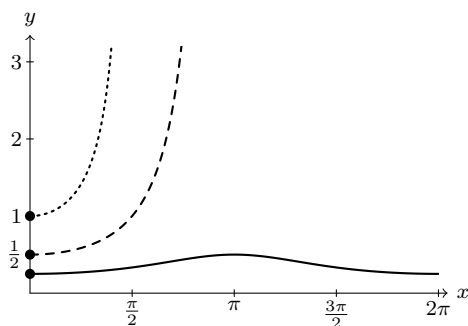
Its derivative is $a^2 \sin x / (1 - a + a \cos x)^2 = (\sin x)y^2$, and $y(0) = a$.

Step 2: Classify all positive parameters. The denominator $D = 1 - a + a \cos x$ ranges from $1 - 2a$ to 1.

- If $0 < a < 1/2$, then $D > 0$ everywhere: $I = \mathbb{R}$ and y is bounded.
- If $a = 1/2$, then $y = 1/(1 + \cos x)$ and $I = (-\pi, \pi)$.
- If $a > 1/2$, let $\alpha = \arccos(1 - 1/a) \in (0, \pi)$. Then $I = (-\alpha, \alpha)$.

In the last two cases $D \downarrow 0$ at either endpoint from inside I , so $y \rightarrow +\infty$. Thus the threshold itself already blows up.

Step 3: Determine extrema and audit the return claim. For $0 < a < 1/2$, $\min y = a$ at $x = 2n\pi$ and $\max y = a/(1 - 2a)$ at $x = (2n + 1)\pi$, $n \in \mathbb{Z}$; 2π is a period. Only this parameter range gives a solution through the full cycle. For $a \geq 1/2$, a pole occurs before reaching 2π . Substituting 2π into the algebraic formula gives a value on a disconnected branch, not a continuation of the initial solution.



Solid: $a = 1/4$. Dashed: $a = 1/2$. Dotted: $a = 1$. Only the solid curve completes a cycle.

Question 9

Let A and B be prescribed real numbers. Seek a real function y that is continuously differentiable on $[-1, 1]$ and satisfies

$$y' = 2xy^2, \quad y(-1) = A, \quad y(1) = B.$$

At the endpoints, derivatives are understood as one-sided derivatives.

Tasks

1. For $A \neq 0$, separate the equation and express the candidate solution in terms of A . Also treat $A = 0$.
2. Determine all pairs (A, B) for which a solution exists on the entire interval, and prove your conditions are sufficient as well as necessary.
3. Show why matching the two endpoint values in a formula does not by itself prove existence. Use $A = B = -2$ as a test case.
4. For every admissible pair, find $y(0)$ and decide whether the solution is unique.

Question 9 – Solution

Strategy. Separate from the left endpoint, then check the denominator on the whole prescribed interval before accepting the right endpoint.

Step 1: Obtain the candidate. For a nonzero solution,

$$-\frac{1}{y} = x^2 + C, \quad \frac{1}{y} = \frac{1}{A} + 1 - x^2, \quad \boxed{y = \frac{A}{1 + A(1 - x^2)}}.$$

For $A = 0$, $y \equiv 0$ is a solution. No solution can leave or cross zero: the right-hand side $2xy^2$ has a continuous y -derivative, $4xy$, so the usual local uniqueness theorem applies at every finite point. Consequently the separated formula applies to any solution with $A \neq 0$ throughout its interval.

Step 2: Test the entire interval. On $[-1, 1]$, the quantity $1 - x^2$ ranges over $[0, 1]$. If $A \geq 0$, the denominator is at least 1. If $-1 < A < 0$, it is at least $1 + A > 0$. For $A = -1$ it vanishes at $x = 0$; for $A < -1$ it vanishes at

$$x = \pm\sqrt{1 + 1/A} \in (-1, 1).$$

The nonzero numerator cannot cancel these zeros. Any admissible formula gives $y(1) = A$, so the necessary conditions are

$$\boxed{B = A \quad \text{and} \quad A > -1}.$$

They are also sufficient: the displayed formula, including $A = 0$, is smooth on a neighborhood of $[-1, 1]$. Its derivative is $2xA^2/[1 + A(1 - x^2)]^2 = 2xy^2$, and it gives both specified endpoint values.

Step 3: Expose the false endpoint test. For $A = B = -2$, the candidate is $y = -2/(2x^2 - 1)$. It has the desired values at $x = \pm 1$ but poles at $x = \pm 1/\sqrt{2}$. Hence it is not a solution on the entire interval.

Step 4: Give the center value and uniqueness. For every admissible pair, $\boxed{y(0) = A/(1 + A)}$. The initial value at -1 already fixes the candidate uniquely; for $A = 0$, local uniqueness also forbids any departure from zero. The second endpoint is a compatibility condition, not an extra free integration constant.

Question 10

Design an affine coefficient $g(x) = \alpha + \beta x$ in the separable equation

$$y' = g(x)(1 - y^2)$$

so that one solution satisfies

$$y(0) = 0, \quad y(1) = \frac{1}{2}, \quad y'(1) = 0.$$

The constants α and β are real.

Tasks

1. Find α and β and justify why the three requirements determine them uniquely.
2. Find an explicit formula for the selected solution. You may use exponentials instead of hyperbolic functions.
3. Determine its maximal interval, every zero, its global maximum, and its limits as $x \rightarrow \pm\infty$.
4. Verify the three requirements directly and list the constant solutions omitted when separating by $1 - y^2$.

Question 10 – Solution

Strategy. Use the zero terminal slope to constrain the coefficient, then use the separated integral to enforce the target height.

Step 1: Determine the coefficient. Since $1 - y(1)^2 = 3/4$, $y'(1) = 0$ forces $\alpha + \beta = 0$. The smooth right-hand side gives local uniqueness, so the solution starting at 0 cannot meet either constant solution $y = \pm 1$. On its branch $-1 < y < 1$,

$$\frac{1}{2} \ln \frac{1+y}{1-y} = \alpha x + \frac{\beta x^2}{2}.$$

At $x = 1$, this becomes $\frac{1}{2} \ln 3 = \alpha + \beta/2$. Hence

$$\boxed{\alpha = \ln 3, \quad \beta = -\ln 3}.$$

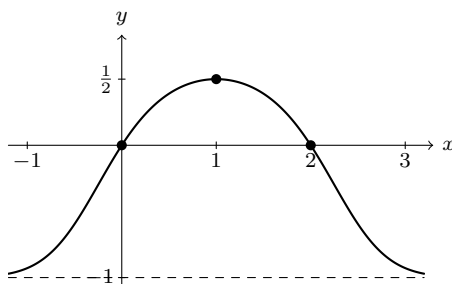
These two independent linear conditions have exactly one pair of coefficients.

Step 2: Solve and interpret the curve. Set $E(x) = 3^{2x-x^2} > 0$. Exponentiating and isolating y gives

$$\boxed{y(x) = \frac{E(x) - 1}{E(x) + 1}, \quad I = \mathbb{R}}.$$

Since $E + 1 > 0$, the formula is globally smooth and lies strictly between -1 and 1 . Its zeros are $x = 0, 2$. The exponent $2x - x^2 = 1 - (x - 1)^2$ has its unique maximum at 1 , and $(E - 1)/(E + 1)$ increases with E . Thus the unique global maximum is $y(1) = 1/2$, while both limits at infinity are -1 .

Step 3: Verify and restore constants. Here $E' = 2(\ln 3)(1 - x)E$ and $1 - y^2 = 4E/(E + 1)^2$, so $y' = (\ln 3)(1 - x)(1 - y^2)$. Also $E(0) = 1$ and $E(1) = 3$ verify both values and the zero slope. The omitted constants are $\boxed{y \equiv 1, y \equiv -1}$; neither satisfies $y(0) = 0$.



The unique maximum is $(1, 1/2)$; both tails approach the dashed level $y = -1$.