

Question 1

Consider the initial value problem

$$(1 + x^2)y' + 2xy = 3x^2 + 1, \quad y(0) = 2.$$

Tasks

1. Put the equation in standard linear form $y' + p(x)y = q(x)$ and find an integrating factor. Explain why multiplying by that factor produces a product derivative.
2. Find the general solution and then the solution of the IVP.
3. Verify the IVP solution in the original, unnormalized equation and state its largest open interval.
4. Identify a straight-line particular solution and determine whether every solution approaches that line as $x \rightarrow +\infty$ and as $x \rightarrow -\infty$. State precisely what approaches zero.

Question 1 – Solution

Strategy. Normalize first. Choose μ with $\mu' = p\mu$, so the product rule gives $(\mu y)' = \mu(y' + py)$.

Step 1: Integrating factor. Since $1 + x^2 > 0$,

$$y' + \frac{2x}{1+x^2}y = \frac{3x^2+1}{1+x^2}, \quad \mu = e^{\int 2x/(1+x^2) dx} = 1+x^2.$$

Multiplication restores the original left side as $[(1+x^2)y]'$.

Step 2: Integrate and select the constant. We obtain

$$(1+x^2)y = \int (3x^2+1) dx = x^3 + x + C,$$

so

$$y = x + \frac{C}{1+x^2}.$$

At $x = 0$, $C = 2$. The IVP solution is therefore $y = x + 2/(1+x^2)$.

Step 3: Direct check and interval. Its derivative is $y' = 1 - 4x/(1+x^2)^2$, and

$$\begin{aligned}(1+x^2)y' + 2xy &= 1+x^2 - \frac{4x}{1+x^2} + 2x^2 + \frac{4x}{1+x^2} \\ &= 3x^2 + 1.\end{aligned}$$

Also $y(0) = 2$. No denominator vanishes for real x , so the largest open interval is $\mathbb{R}$.

Step 4: Approach to a line. Setting $C = 0$ gives the particular solution $y_p = x$. For every fixed real C ,

$$y(x) - x = \frac{C}{1+x^2} \longrightarrow 0 \quad \text{as } x \rightarrow \pm\infty.$$

It is the vertical difference from the line that tends to zero; the solution itself does not tend to a finite constant.

Question 2

On the domain $x > 0$, consider

$$xy' + 2y = x^2 \ln x, \quad y(1) = 0.$$

Tasks

1. Solve the IVP using an integrating factor, showing the integration-by-parts step needed for the forcing term.
2. State the largest open interval containing 1 on which the solution solves this equation, and verify the initial value and differential equation.
3. Rewrite the solution using a definite integral based at 1. Use the sign of that integral to decide whether the solution can be negative.
4. Locate the global minimum on $x > 0$ and sketch the solution near $x = 1$ in your solution. Explain why positive forcing only for $x > 1$ does not make the solution negative for $0 < x < 1$.

Question 2 – Solution

Strategy. Use x^2 as integrating factor, then retain a definite-integral form to analyze signs on both sides of the initial point.

Step 1: Solve. Dividing by x gives $y' + 2y/x = x \ln x$. With $\mu = x^2$,

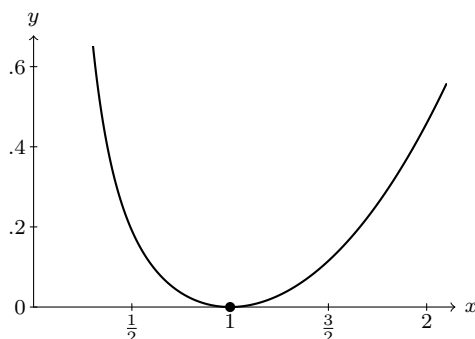
$$(x^2 y)' = x^3 \ln x.$$

Integration by parts, $u = \ln x$, $dv = x^3 dx$, yields

$$\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \frac{x^4}{16} + C.$$

The initial value forces $C = 1/16$, hence

$$y = \frac{x^2}{4} \ln x - \frac{x^2}{16} + \frac{1}{16x^2}, \quad x > 0.$$



The solution is positive except at its global minimum $(1, 0)$.

Step 2: Verify. Substitution at 1 gives zero. Differentiating $x^2 y$ gives $x^3 \ln x$, or $x^2 y' + 2xy = x^3 \ln x$; division by $x > 0$ recovers the original equation. The logarithm and coefficient division restrict the interval to $(0, \infty)$.

Step 3: Sign and minimum. Equivalently,

$$x^2 y(x) = \int_1^x t^3 \ln t \, dt.$$

For $x > 1$ the integral is positive. For $0 < x < 1$, the integrand is negative between x and 1, but the reversed limits make the integral positive. Since $x^2 > 0$, $y(x) > 0$ for $x \neq 1$. Thus $\min_{x>0} y = 0$ only at $x = 1$. The sign of the forcing alone does not determine the sign of the solution.

Question 3

A student tries to solve

$$y' - 2y = xe^{2x}, \quad y(0) = 0,$$

and writes

$$\mu = e^{2x}, \quad (e^{2x}y)' = xe^{4x}.$$

Tasks

1. Identify the precise error by expanding the student's claimed product derivative.
2. Find a correct integrating factor and solve the IVP.
3. Verify the solution by direct substitution in $y' - 2y = xe^{2x}$.
4. Replace the initial condition by $y(0) = a$, with a any real number. Find the resulting family and prove directly that two different choices of a cannot produce intersecting solution curves.

Question 3 – Solution

Strategy. The integrating factor must match the signed coefficient of y . Check its product derivative before integrating.

Step 1: Diagnose the sign error. The student's derivative is

$$(e^{2x}y)' = e^{2x}(y' + 2y),$$

whereas multiplying the original equation by e^{2x} gives $e^{2x}(y' - 2y)$. These expressions differ in the sign of the y term. The required coefficient is $p = -2$, so $\mu = e^{-2x}$.

Step 2: Correct integration. Multiplication gives

$$(e^{-2x}y)' = x, \quad e^{-2x}y = \frac{x^2}{2} + C.$$

Since $y(0) = 0$, $C = 0$, and

$$y = \frac{x^2}{2}e^{2x}, \quad x \in \mathbb{R}.$$

Step 3: Verify. Differentiating explicitly,

$$y' = xe^{2x} + x^2e^{2x}, \quad y' - 2y = xe^{2x} + x^2e^{2x} - x^2e^{2x} = xe^{2x}.$$

The initial value is also zero.

Step 4: Vary the initial datum. The general initial value sets $C = a$, giving

$$y_a(x) = \left(a + \frac{x^2}{2}\right)e^{2x}.$$

For distinct a, b ,

$$y_a(x) - y_b(x) = (a - b)e^{2x} \neq 0$$

at every real x . Hence their graphs never intersect. The nonintersection follows from the explicit homogeneous difference, not merely from a sketch.

Question 4

Consider the linear equation

$$y' - y = e^{-x}, \quad x \geq 0.$$

Instead of prescribing an initial value, require that the solution remain bounded on the entire half-line $[0, \infty)$.

Tasks

1. Find the general solution using an integrating factor.
2. Determine the unique bounded solution and the initial value it must have.
3. If that initial value is changed by a nonzero amount ε , find the difference from the bounded solution and describe its behavior as $x \rightarrow \infty$.
4. Sketch the bounded solution and the solutions for $\varepsilon = \pm 1/20$ in your solution. Explain why a forcing term tending to zero does not force every solution to tend to zero.

Question 4 – Solution

Strategy. Separate the forced response from the homogeneous term. A condition at infinity can select the constant just as an initial value can.

Step 1: General solution. With $\mu = e^{-x}$,

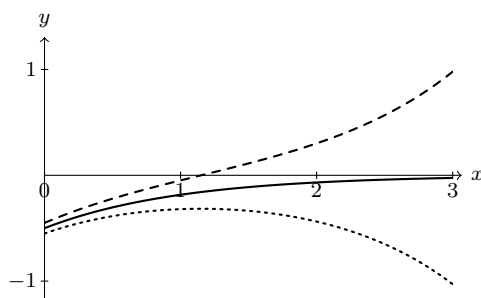
$$(e^{-x}y)' = e^{-2x}, \quad e^{-x}y = -\frac{1}{2}e^{-2x} + C.$$

Thus $y = Ce^x - \frac{1}{2}e^{-x}$; each formula is defined on $\mathbb{R}$.

Step 2: Impose boundedness. As $x \rightarrow \infty$, the second term tends to zero. If $C \neq 0$, the term Ce^x is unbounded and cannot be canceled by that decaying term. Therefore

$$y_b = -\frac{1}{2}e^{-x}, \quad y_b(0) = -\frac{1}{2}.$$

This solution is bounded and tends to zero, proving both existence and uniqueness under the stated boundedness condition.



Solid: bounded solution. Dashed: $\varepsilon = 1/20$. Dotted: $\varepsilon = -1/20$.

Step 3: Perturb the selected initial value. Since $y(0) = C - 1/2$, changing it to $-1/2 + \varepsilon$ gives $C = \varepsilon$. Consequently

$$y_\varepsilon - y_b = \varepsilon e^x.$$

For $\varepsilon > 0$ the perturbed solution tends to $+\infty$; for $\varepsilon < 0$ it tends to $-\infty$. Direct differentiation verifies $y' - y = e^{-x}$ for every C . The vanishing forcing does not remove a growing homogeneous component.

Question 5

Two functions are known to solve the same first-order equation in standard linear form,

$$y' + p(x)y = q(x), \quad x \in \mathbb{R},$$

where p, q are continuous. The functions are

$$y_1 = x + e^{-x}, \quad y_2 = x - e^{-x}.$$

Tasks

1. Recover $p(x)$ and $q(x)$ from these two solutions. Explain why they are determined at every real x .
2. Solve the recovered equation using an integrating factor to obtain all its solutions.
3. Determine all real constants α, β for which $\alpha y_1 + \beta y_2$ is also a solution of the same nonhomogeneous equation.
4. Explain why arbitrary sums of solutions need not remain solutions, although the difference of two solutions always satisfies the associated homogeneous equation.

Question 5 – Solution

Strategy. Subtract the two equations to eliminate the forcing. A nonzero solution difference reveals the coefficient p .

Step 1: Recover the coefficients. Let $d = y_1 - y_2 = 2e^{-x}$. Subtraction gives $d' + pd = 0$, so

$$p = -\frac{d'}{d} = 1,$$

valid for every real x because d never vanishes. Using $y_1' = 1 - e^{-x}$,

$$q = y_1' + y_1 = 1 - e^{-x} + x + e^{-x} = x + 1.$$

Thus $\boxed{y' + y = x + 1}$ is uniquely recovered within the stated standard linear form.

Step 2: All solutions. The integrating factor is e^x , giving

$$(e^x y)' = (x + 1)e^x = (xe^x)'.$$

Integration yields $\boxed{y = x + Ce^{-x}, \quad C \in \mathbb{R}}$. Substitution confirms $y' + y = x + 1$ for every constant.

Step 3: Which combinations work? Write $L[v] = v' + v$. Linearity gives

$$L[\alpha y_1 + \beta y_2] = (\alpha + \beta)(x + 1).$$

For this to equal $x + 1$ throughout an interval, we require

$$\boxed{\alpha + \beta = 1.}$$

These combinations are $x + (\alpha - \beta)e^{-x}$ and generate every solution.

Step 4: Nonhomogeneous versus homogeneous. The sum $y_1 + y_2 = 2x$ produces $L[2x] = 2x + 2$, not $x + 1$. The difference instead satisfies $L[y_1 - y_2] = q - q = 0$. Thus differences solve the homogeneous equation; combinations with coefficients summing to one preserve the original forcing.

Question 6

For each real initial value a , consider

$$y' + 2xy = 1, \quad y(0) = a.$$

A definite integral is an acceptable exact expression; no elementary antiderivative is required.

Tasks

1. Use an integrating factor to obtain an exact solution based at $x = 0$, and state its interval of definition.
2. Verify the formula by differentiation and the initial condition directly.
3. For solutions with initial values a and b , determine their difference, its greatest absolute value on $\mathbb{R}$, and whether their curves can cross when $a \neq b$.
4. Show that the solution with $a = 0$ is odd and positive for $x > 0$. Sketch the curves for $a = -1, 0, 1$ in your solution.

Question 6 – Solution

Strategy. Use a definite integral to incorporate the initial value exactly. The homogeneous difference makes comparison easier than evaluating the integral.

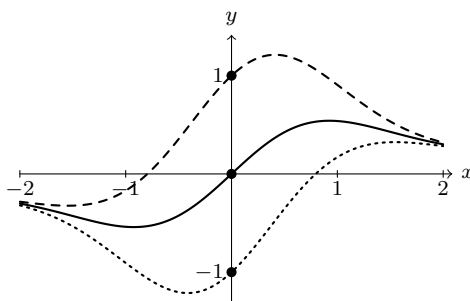
Step 1: Integrating factor and solution. Here

$$\mu = e^{\int 2x \, dx} = e^{x^2}, \quad (e^{x^2} y)' = e^{x^2}.$$

Integrating from 0 to x gives

$$y_a(x) = e^{-x^2} \left(a + \int_0^x e^{t^2} \, dt \right), \quad x \in \mathbb{R}.$$

The integral exists for every finite real x .



Solid: $a = 0$. Dashed: $a = 1$. Dotted: $a = -1$. The curves never cross.

Step 2: Direct verification. The product rule and Fundamental Theorem give $y'_a = -2xy_a + e^{-x^2}e^{x^2} = 1 - 2xy_a$. Also the integral is zero at 0, so $y_a(0) = a$.

Step 3: Compare initial values. The integral cancels in the difference:

$$y_a - y_b = (a - b)e^{-x^2}, \quad \max_{x \in \mathbb{R}} |y_a - y_b| = |a - b|.$$

For $a \neq b$, equality in the maximum occurs only at $x = 0$, and the difference never changes sign or vanishes at a finite point. The curves cannot cross.

Step 4: Symmetry of the zero-data solution. Since e^{t^2} is even, $\int_0^x e^{t^2} \, dt$ is odd. Multiplying by the even factor e^{-x^2} preserves oddness. For $x > 0$, both the integral and prefactor are positive, so $y_0(x) > 0$.

Question 7

Consider the periodically forced equation

$$y' + 2y = 3 \cos x + \sin x,$$

with initial condition $y(0) = 0$.

Tasks

1. Solve the IVP using an integrating factor. Show how you evaluate the integral of the forcing times that factor.
2. Find the unique 2π -periodic solution of the equation, and verify both its periodicity and its differential equation.
3. Find the earliest $X \geq 0$ such that the IVP solution differs from the periodic solution by at most $1/100$ for every $x \geq X$.
4. Sketch the IVP solution and the periodic solution on $[0, 2\pi]$ in your solution. Explain which part of the general solution is the transient and why the prescribed IVP solution is not itself periodic.

Question 7 – Solution

Strategy. Integrate the forcing after multiplying by e^{2x} , then separate periodic and decaying terms.

Step 1: Integrate. The equation becomes

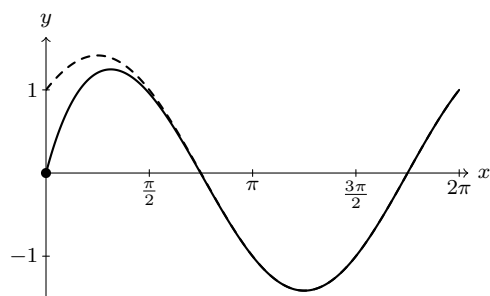
$$(e^{2x}y)' = e^{2x}(3\cos x + \sin x).$$

Seek an antiderivative $e^{2x}(A\cos x + B\sin x)$. Matching coefficients gives $2A + B = 3$ and $2B - A = 1$, hence $A = B = 1$. Indeed,

$$[e^{2x}(\cos x + \sin x)]' = e^{2x}(3\cos x + \sin x).$$

Therefore $y = \cos x + \sin x + Ce^{-2x}$. Since $y(0) = 0$, $C = -1$, giving

$$y = \cos x + \sin x - e^{-2x}.$$



Solid: IVP solution. Dashed: periodic solution $y_p = \cos x + \sin x$.

Step 2: The only periodic member. The function $y_p = \cos x + \sin x$ is 2π -periodic and satisfies $y_p' + 2y_p = 3\cos x + \sin x$. For the general solution,

$$y(x + 2\pi) - y(x) = Ce^{-2x}(e^{-4\pi} - 1).$$

This vanishes for every x only if $C = 0$. Hence $y_p = \cos x + \sin x$ is the unique periodic member; the IVP has $C = -1$ and is not periodic.

Step 3: Uniform accuracy after a time. The exact discrepancy is $|y - y_p| = e^{-2x}$, strictly decreasing on $x \geq 0$. Thus

$$e^{-2X} \leq \frac{1}{100} \iff X \geq \ln 10.$$

The earliest such time is $\ln 10$. The term Ce^{-2x} is transient because it vanishes as $x \rightarrow \infty$, leaving the periodic response.

Question 8

A switched forcing acts on a linear equation for $t \geq 0$:

$$y' + y = f(t), \quad y(0) = 0, \quad f(t) = \begin{cases} 2, & 0 \leq t < 1, \\ 0, & t > 1. \end{cases}$$

Seek a continuous function that is continuously differentiable on each side of $t = 1$ and satisfies the equation away from that switching time. No derivative at $t = 1$ is assumed.

Tasks

1. Solve on $0 \leq t < 1$ using an integrating factor, then use continuity to solve on $t > 1$.
2. Find the global maximum for $t \geq 0$, including its time, and the limit as $t \rightarrow \infty$.
3. Compute the left and right derivatives at $t = 1$. Is the resulting function a classical differentiable solution at that point?
4. Evaluate $\int_0^\infty y(t) dt$ and relate it to the total forcing $\int_0^\infty f(t) dt$. Sketch the continuous response in your solution, showing the change in slope at the switch.

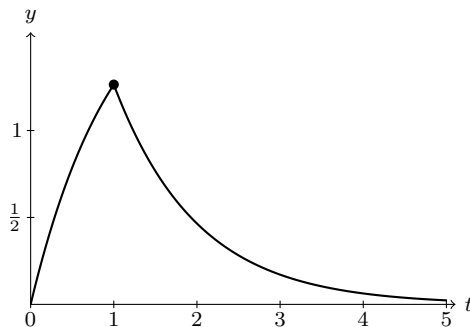
Question 8 – Solution

Strategy. Integrate separately on the two smooth pieces and match values at the switch; continuity need not imply a continuous derivative.

Step 1: Solve and match. Before the switch, $(e^t y)' = 2e^t$, so $y = 2 + Ce^{-t}$. Initial data give $C = -2$ and $y(1) = A := 2(1 - e^{-1})$. Afterward $(e^t y)' = 0$; matching this value yields

$$y(t) = \begin{cases} 2(1 - e^{-t}), & 0 \leq t \leq 1, \\ Ae^{-(t-1)}, & t > 1, \end{cases} \quad A = 2(1 - e^{-1}).$$

Differentiation on each open piece verifies its corresponding equation.



The maximum occurs at the switch $t = 1$, where the derivative jumps by -2 .

Step 2: Maximum and derivative jump. Before 1, $y' = 2e^{-t} > 0$; after 1, $y' = -Ae^{-(t-1)} < 0$. Hence the global maximum is A at $t = 1$, and the limit is zero. At the switch,

$$y'(1^-) = 2/e, \quad y'(1^+) = -A = 2/e - 2.$$

The jump is -2 , so the derivative does not exist there. The response meets the requested piecewise conditions, but is not a classical differentiable solution at the switch.

Step 3: Total response. Direct integration gives

$$\int_0^1 2(1 - e^{-t})dt = 2/e, \quad \int_1^\infty Ae^{-(t-1)}dt = A,$$

so $\int_0^\infty y(t)dt = 2 = \int_0^\infty f(t)dt$. Equivalently, integrate $y' + y = f$ on the two pieces: continuity cancels the joining values and $y(0) = \lim y = 0$ cancels the endpoint contribution.

Question 9

The equation

$$xy' - y = x^2$$

has a vanishing leading coefficient at $x = 0$. Initially impose $y(1) = 2$.

Tasks

1. Solve the IVP on $x > 0$ using an integrating factor after normalization.
2. Determine whether this solution extends to a differentiable solution of the original equation on all of $\mathbb{R}$. If so, prove that such an extension is unique.
3. Now discard $y(1) = 2$ and prescribe only $y(0) = 0$. Determine all differentiable solutions on $\mathbb{R}$ satisfying this new initial condition.
4. Decide whether a differentiable solution with $y(0) = 1$ can exist. Explain why dividing by x and invoking a regular standard-form IVP at 0 would be invalid.

Question 9 – Solution

Strategy. Normalize only where $x \neq 0$. To cross the exceptional point, return to the original equation and check value and derivative matching.

Step 1: Solve on the positive half-line. Dividing by $x > 0$ gives $y' - y/x = x$. The integrating factor is $e^{-\ln x} = 1/x$, so

$$(y/x)' = 1, \quad y = x^2 + Cx.$$

Since $y(1) = 2$, $C = 1$, giving $y = x^2 + x \quad (x > 0)$.

Step 2: Unique differentiable extension. The polynomial $x^2 + x$ satisfies $xy' - y = x^2$ everywhere, including $x = 0$. Any solution on $x < 0$ also satisfies $(y/x)' = 1$, hence has form $x^2 + Dx$. Continuity at 0 forces $y(0) = 0$, and differentiability requires the slopes at zero to match: $D = 1$. Thus

$$y = x^2 + x \text{ is the unique differentiable extension to } \mathbb{R}.$$

Step 3: Data only at the exceptional point. With only $y(0) = 0$, the same argument allows any common slope C on the two sides. All global differentiable solutions are

$$y = x^2 + Cx, \quad C \in \mathbb{R}.$$

Indeed $x(2x + C) - (x^2 + Cx) = x^2$, and every such polynomial has value zero at 0. This IVP is therefore nonunique.

Step 4: Incompatible data and normalization. At $x = 0$, the original equation requires $-y(0) = 0$, ruling out $y(0) = 1$. The normalized coefficient $-1/x$ is undefined at zero, so regular standard-form conclusions cannot be applied there. The algebraic condition at the exceptional point must be checked separately.

Question 10

Let a be a real parameter in

$$y' + ay = e^{-x}, \quad y(0) = 1.$$

Tasks

1. Solve the IVP for every real a , handling separately any parameter value for which the usual antiderivative formula changes.
2. Verify the initial condition and equation, and state the solution interval for each parameter.
3. For fixed x , show that the formula for $a \neq 1$ tends to the $a = 1$ solution as $a \rightarrow 1$. Explain why an apparent denominator singularity in a parameter need not be a singularity of the solution.
4. Determine exactly which values of a produce a bounded solution on $[0, \infty)$, and compute the limit or divergence as $x \rightarrow \infty$ in each parameter range.

Question 10 – Solution

Strategy. Keep the parameter visible during integration, and isolate the exceptional exponent before dividing by $a - 1$.

Step 1: Integrate with the initial value. With $\mu = e^{ax}$,

$$(e^{ax}y)' = e^{(a-1)x}, \quad e^{ax}y = 1 + \int_0^x e^{(a-1)t} dt.$$

Consequently

$$y(x) = \begin{cases} \frac{e^{-x} + (a-2)e^{-ax}}{a-1}, & a \neq 1, \\ (1+x)e^{-x}, & a = 1. \end{cases}$$

Each formula is defined on $\mathbb{R}$.

Step 2: Verify. At zero the numerator for $a \neq 1$ is $a - 1$, so $y(0) = 1$; the exceptional formula also gives 1. Differentiating the definite-integral identity above gives $y' + ay = e^{-x}$. At $a = 1$, a direct check is $y' = -xe^{-x}$ and $y' + y = e^{-x}$.

Step 3: Remove the apparent parameter singularity. For $a \neq 1$, rewrite the result as

$$y = e^{-ax} \left[1 + \frac{e^{(a-1)x} - 1}{a-1} \right].$$

For fixed x , the quotient tends to x as $a \rightarrow 1$, by the derivative of the exponential (also at $x = 0$). Hence $y \rightarrow (1+x)e^{-x}$. The singularity in the divided parameter formula is removable.

Step 4: Long-time behavior. If $a > 0$ and $a \neq 1$, both exponentials decay, so $y \rightarrow 0$; the same holds at $a = 1$. If $a = 0$, $y = 2 - e^{-x} \rightarrow 2$. If $a < 0$, the coefficient $(a-2)/(a-1)$ is positive and e^{-ax} grows without bound, so $y \rightarrow +\infty$. Thus

$$y \text{ is bounded on } [0, \infty) \iff a \geq 0.$$