

Question 1

Consider the initial value problem

$$y' = e^{-x^2}, \quad y(0) = 1.$$

A proposed answer is

$$F(x) = 1 + \int_0^x e^{-t^2} dt.$$

A student rejects this expression because the integral has no elementary antiderivative. You may take that non-elementarity as given; proving it is not part of this problem.

Tasks

1. Verify F as an exact solution and state its interval of definition.
2. Prove uniqueness directly: if G is another solution with $G(0) = 1$ on an interval containing 0, show that $G = F$ there.
3. Without evaluating the integral numerically, prove the bounds

$$1 + e^{-1} < F(1) < 2.$$

4. Explain the distinction between an exact representation, an elementary formula, and a numerical approximation. Does the lack of an elementary antiderivative imply that the IVP has no solution?

Question 1 – Solution

Strategy. A definite integral defines a function. The Fundamental Theorem of Calculus can verify that function even when elementary antidifferentiation is unavailable.

Step 1: Verification and domain. The integrand is continuous for every real t , so the integral exists for every finite real x . The Fundamental Theorem gives

$$F'(x) = e^{-x^2}, \quad F(0) = 1 + 0 = 1.$$

Thus $\boxed{F \text{ is an exact solution on } \mathbb{R}.}$ Negative x causes no problem: the integral is interpreted with reversed limits.

Step 2: Uniqueness without a general theorem. If G solves the same IVP on an interval I containing 0, then

$$(G - F)' = e^{-x^2} - e^{-x^2} = 0.$$

A differentiable function with zero derivative on an interval is constant there. At 0, $G - F = 0$, so

$$\boxed{G(x) = F(x) \quad (x \in I).}$$

Step 3: Strict integral bounds. For $0 < t < 1$,

$$e^{-1} < e^{-t^2} < 1.$$

These strict inequalities hold throughout an interval of positive length. Integrating from 0 to 1 therefore gives

$$e^{-1} < \int_0^1 e^{-t^2} dt < 1, \quad \boxed{1 + e^{-1} < F(1) < 2.}$$

Step 4: What counts as a solution. The definite-integral expression specifies F exactly, with no rounding. An elementary formula is a more restricted type of expression, assembled from familiar elementary functions. A decimal obtained by numerical integration is an approximation to a value of F . Failure to find an elementary expression does not negate the exact solution already verified.

Question 2

The exact solution of $y' = y$, $y(0) = 1$, is supplied as $y = e^x$. A polynomial approximation is

$$P(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

You may use Taylor's theorem with a fourth-derivative remainder.

Tasks

1. Calculate the residual $P' - P$ and determine whether P solves the IVP exactly on any open interval containing 0.
2. Let $E(x) = e^x - P(x)$. Prove that for $0 \leq x \leq 1$,

$$\frac{x^4}{24} \leq E(x) \leq \frac{ex^4}{24}.$$

3. Give exact lower and upper bounds for $E(1)$, and identify whether P overestimates or underestimates the solution for $0 < x \leq 1$.
4. Plot the error and its two bounds in your solution. Explain why matching the initial value and several derivatives at 0 does not make the polynomial an exact interval solution.

Question 2 – Solution

Strategy. Use a residual to test exactness and Taylor's theorem to assess the approximation. These answer different questions.

Step 1: Exactness test. Since $P' = 1 + x + x^2/2$,

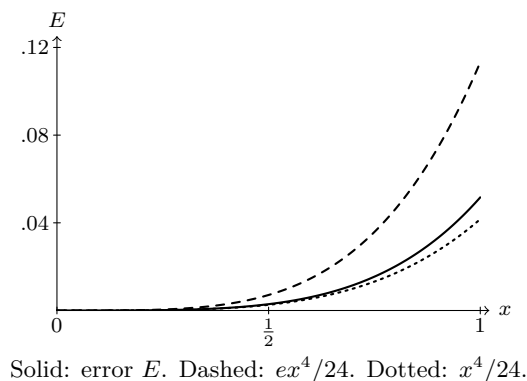
$$P' - P = -\frac{x^3}{6}.$$

The residual is nonzero whenever $x \neq 0$. Although $P(0) = 1$, the polynomial does not solve the equation on any open interval containing 0.

Step 2: Error bounds. For $0 < x \leq 1$, Taylor's theorem supplies a number ξ with $0 < \xi < x$ such that

$$E(x) = \frac{e^\xi x^4}{24}.$$

Because $1 \leq e^\xi \leq e$, the requested bounds follow; at $x = 0$ all three expressions are zero.



Step 3: Size and sign. At $x = 1$, $P(1) = 8/3$, so

$$\frac{1}{24} \leq E(1) = e - \frac{8}{3} \leq \frac{e}{24}.$$

For $x > 0$ the lower bound is positive, so P underestimates the exact solution. The two endpoint bounds are in fact strict at 1.

Step 4: Local agreement is limited evidence. The values of P and its first three derivatives agree with those of e^x at 0. That is local Taylor information. Solving the ODE requires $P'(x) = P(x)$ at every point of an interval, which the nonzero residual disproves.

Question 3

Two measurements of an initial value lead to the candidate functions

$$y(x) = e^{2x}, \quad \tilde{y}(x) = (1 + \varepsilon)e^{2x}, \quad \varepsilon = 10^{-3},$$

for the equation $y' = 2y$. Define absolute error and relative error by

$$A(x) = |\tilde{y}(x) - y(x)|, \quad R(x) = \frac{|\tilde{y}(x) - y(x)|}{|y(x)|}.$$

Tasks

1. Verify both candidates and identify their different initial values.
2. Determine $A(x)$ and $R(x)$ for $x \geq 0$. Find the first time the absolute error reaches 0.1.
3. For the equation $z' = -2z$, compare the candidates $z = e^{-2x}$ and $\tilde{z} = (1 + \varepsilon)e^{-2x}$ in the same way.
4. Plot the two absolute errors in your solution. Explain why increasing separation here is not evidence that one IVP has multiple solutions, and why absolute and relative error tell different stories.

Question 3 – Solution

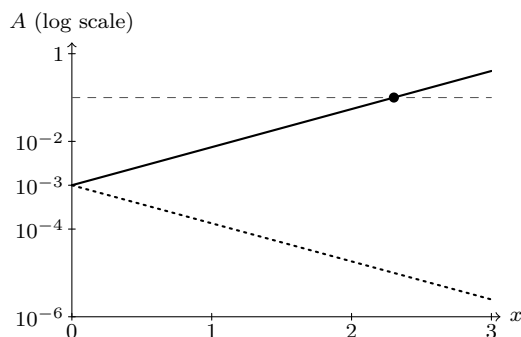
Strategy. Keep the initial conditions distinct and compute both error measures before interpreting the behavior.

Step 1: Verification. Differentiation gives $y' = 2y$ and $\tilde{y}' = 2\tilde{y}$. Their initial values are 1 and 1.001, respectively. They solve different IVPs for the same differential equation.

Step 2: Growth case. For $x \geq 0$,

$$A(x) = 10^{-3}e^{2x}, \quad R(x) = 10^{-3}.$$

The absolute error increases strictly. Its first value 0.1 occurs when $e^{2x} = 100$, giving $x = \ln 10$.



Solid: growth error. Dotted: decay error. Dashed: $A = 0.1$; dot at $x = \ln 10$.

Step 3: Decay case. Both decay candidates satisfy $z' = -2z$, again with initial values 1 and 1.001. Their errors are

$$|\tilde{z} - z| = 10^{-3}e^{-2x}, \quad \frac{|\tilde{z} - z|}{|z|} = 10^{-3}.$$

The absolute error decreases to zero and never reaches 0.1 for $x \geq 0$. The relative error stays fixed at 0.1% in both examples.

Step 4: Interpretation. Uniqueness concerns identical equations and identical initial data. Here the data differ. In the growth case the absolute discrepancy becomes large, yet it remains the same fraction of the growing reference solution. In the decay case both the solution and its discrepancy shrink proportionally.

Question 4

For the initial value problem

$$y' = y^2, \quad y(0) = 1,$$

a candidate is

$$f(x) = \frac{1}{1-x}.$$

A report claims: “The right-hand side y^2 is smooth everywhere, so this formula gives a solution through $x = 1$ and for all real x .”

Tasks

1. Verify the candidate wherever it is defined. Find the largest open interval containing 0 on which it solves the IVP.
2. Calculate its one-sided limits at $x = 1$ and explain why assigning any finite value at 1 cannot extend this solution across that point.
3. Does the same formula solve the differential equation on $(1, \infty)$? Explain why that branch is not a continuation of the given IVP solution.
4. Sketch both branches in your solution and correct the report’s inference about smooth equations and global solutions.

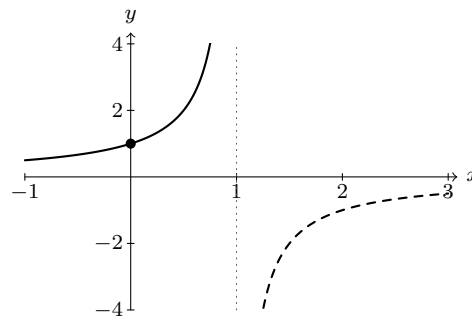
Question 4 – Solution

Strategy. Check the function's domain as well as the equation's right-hand side. A solution can become unbounded in finite time even for a smooth equation.

Step 1: Verification and interval. For $x \neq 1$,

$$f'(x) = \frac{1}{(1-x)^2} = f(x)^2, \quad f(0) = 1.$$

The largest open interval containing 0 that avoids 1 is $(-\infty, 1)$.



Solid: IVP branch. Dashed: separate solution branch. Dotted: $x = 1$.

Step 2: No finite extension. The limits are

$$\lim_{x \rightarrow 1^-} f(x) = +\infty, \quad \lim_{x \rightarrow 1^+} f(x) = -\infty.$$

A differentiable extension across 1 would be continuous there and would have a finite value. The left-hand divergence rules this out, independently of what formula is proposed after 1.

Step 3: The separate branch. On $(1, \infty)$, the same derivative calculation still proves $f' = f^2$. This branch is a solution of the differential equation, but its interval does not contain the initial point 0 and cannot be joined through $x = 1$ to the IVP branch.

Step 4: Corrected inference. Smoothness of a right-hand side is not a guarantee of existence for all time. Here the equation is defined at every finite pair (x, y) , but the IVP solution escapes to unbounded y as $x \rightarrow 1^-$. The failure belongs to global continuation of the solution, not to a singularity of y^2 at a finite point.

Question 5

An experiment records the complete trajectory $p(x) = e^{-x}$ for all $x \geq 0$, with $p(0) = 1$. Two proposed models are

$$(A) \quad y' = -y, \quad (B) \quad y' = -y + (y - e^{-x})^2.$$

Tasks

1. Verify that the recorded trajectory satisfies both models.
2. Compare the slopes predicted by the models at $(0, 0)$ and $(0, 2)$. Are the right-hand sides the same function of (x, y) ?
3. Test whether $y \equiv 0$ is an equilibrium solution of each model.
4. Explain why even a complete, noiseless trajectory does not identify an unrestricted differential equation uniquely. Generalize the construction by replacing the added square with a term involving an arbitrary smooth function $H(x, y)$.

Question 5 – Solution

Strategy. A trajectory tests a model only along one curve in the (x, y) -plane. Compare predictions away from that curve.

Step 1: Agreement on the observed curve. Since $p' = -e^{-x} = -p$, model (A) holds. Along $y = p(x)$, the added term in (B) is $(p - e^{-x})^2 = 0$, so (B) also gives $p' = -p$. Both match the entire observation and its initial value exactly.

Step 2: Different off-curve predictions. At $x = 0$, $e^{-x} = 1$. Therefore

Point	Slope from (A)	Slope from (B)
$(0, 0)$	0	1
$(0, 2)$	-2	-1

Thus the models agree on p but are different laws.

Step 3: An equilibrium distinguishes them. For $y \equiv 0$, the derivative is zero. Model (A) assigns zero, so this is an equilibrium. Model (B) assigns $e^{-2x} > 0$, so the same constant function is not a solution of (B).

Step 4: General non-identifiability. For any smooth H , the model

$$\boxed{y' = -y + H(x, y)(y - e^{-x})^2}$$

admits $p = e^{-x}$, because the added term vanishes identically along that curve. Many choices of H change slopes elsewhere. Additional experiments with different initial data or restrictions on the allowed model class are needed to distinguish such laws. Complete data on one trajectory constrain a curve, not the whole plane.

Question 6

Compare an equation with its squared version:

$$(A) \quad y' = -\sqrt{1-y^2}, \quad |y| \leq 1,$$

$$(B) \quad (y')^2 = 1 - y^2.$$

Both are assigned $y(0) = 0$. Consider $u(x) = \sin x$ and $v(x) = -\sin x$.

Tasks

1. Verify that both candidates satisfy (B) and the initial value. Which satisfies (A) on $(-\pi/2, \pi/2)$?
2. State the sign restriction lost by squaring. Give a condition that, together with (B), recovers (A) for real differentiable functions.
3. Explain why $v = -\sin x$ does not satisfy (A) on all of $\mathbb{R}$.
4. Extend the accepted middle branch to a C^1 solution of (A) on all of $\mathbb{R}$ by adjoining constant pieces outside $[-\pi/2, \pi/2]$. Verify the joins and sketch the extension in your solution.

Question 6 – Solution

Strategy. Squaring forgets a sign. Restore that sign when selecting a branch, and check differentiability when extending it.

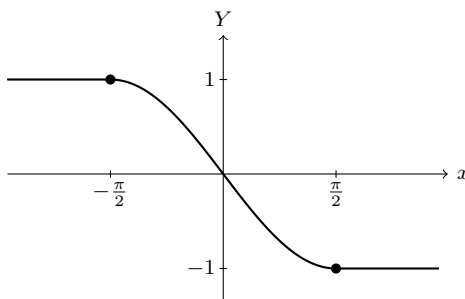
Step 1: Compare the candidates. Both satisfy $(y')^2 = \cos^2 x = 1 - y^2$ and have value zero at 0. On $(-\pi/2, \pi/2)$, $\cos x > 0$, so the right-hand side of (A) is $-|\cos x| = -\cos x$. Hence

$$v = -\sin x \text{ satisfies (A), but } u = \sin x \text{ does not.}$$

Step 2: Recover the lost information. Equation (A) forces $y' \leq 0$. Conversely, (B) and $y' \leq 0$ imply $|y| \leq 1$ and $y' = -\sqrt{1 - y^2}$. Outside the middle interval, $v' = -\cos x$ is positive wherever $\cos x < 0$, so v fails (A) there.

Step 3: A global extension. Define

$$Y(x) = \begin{cases} 1, & x \leq -\pi/2, \\ -\sin x, & -\pi/2 < x < \pi/2, \\ -1, & x \geq \pi/2. \end{cases}$$



The constant pieces meet $-\sin x$ with matching value and zero derivative.

At either join, the middle value matches the adjoining constant and its derivative $-\cos x$ tends to zero, the derivative of the constant piece. Thus Y is C^1 . On the constant pieces and at the joins, both sides of (A) are zero. On the middle piece, the verification above applies. Therefore Y solves the original IVP on $\mathbb{R}$; extension is possible here because both endpoint values and derivatives remain finite.

Question 7

Consider the IVP $y' = 0$, $y(0) = 0$ and the functions

$$u(x) = 0, \quad v(x) = \begin{cases} 0, & x \leq 0, \\ e^{-1/x^2}, & x > 0. \end{cases}$$

You may use the fact that v is infinitely differentiable on $\mathbb{R}$ and that $v^{(n)}(0) = 0$ for every integer $n \geq 0$.

Tasks

1. Determine the Taylor series at 0 for both functions, using the given derivative information.
2. Calculate $v'(x)$ for $x > 0$ and decide whether v solves the IVP on any open interval containing 0.
3. Prove that u is the only solution of the IVP on an interval containing 0.
4. Explain why agreement of every derivative at a single point does not contradict your conclusion. Distinguish a smooth function from one known to equal its Taylor series near that point.

Question 7 – Solution

Strategy. Use interval verification for the ODE. Infinite-order agreement at one point is still pointwise information unless convergence to the function is established.

Step 1: Identical Taylor coefficients. For both functions, every derivative at zero is zero, so both Taylor series are

$$\sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0.$$

This describes the series built from the derivatives; it does not yet assert that the series represents either function away from zero.

Step 2: The equation fails to the right. For $x > 0$, the chain rule gives

$$v'(x) = \frac{2}{x^3} e^{-1/x^2} > 0.$$

Every open interval containing 0 includes positive x , where the required identity $v' = 0$ fails. Thus v does not solve this IVP on such an interval, despite satisfying the initial value and having every derivative there equal to those of u .

Step 3: Uniqueness directly. If $y' = 0$ on an interval, the Mean Value Theorem implies that y is constant on that interval. The initial value forces that constant to be zero. Hence $y \equiv 0$ is the unique IVP solution.

Step 4: Smoothness versus a series representation. Smoothness means the derivatives of all orders exist and are continuous. A function is analytic at a point if it equals its Taylor series in some neighborhood of that point. The supplied v is smooth but not analytic at 0: its Taylor series is zero, whereas $v(x) > 0$ for every $x > 0$. Agreement of derivatives at a single point therefore does not replace the interval identity required by the ODE.

Question 8

An implicit description of a function is proposed for $x \geq 0$:

$$y^3 + xy = 1, \quad y > 0.$$

The associated initial value problem is

$$y' = -\frac{y}{x + 3y^2}, \quad y(0) = 1.$$

You may use implicit differentiation where the coefficient of y' is nonzero.

Tasks

1. Show that for every fixed $x \geq 0$ the implicit relation has exactly one positive root y .
2. Differentiate the relation, verify the ODE and initial value, and calculate the initial slope. Explain why the denominator stays nonzero on this branch.
3. Prove that the branch decreases for $x \geq 0$ and tends to zero as $x \rightarrow \infty$.
4. Prove $2/3 < y(1) < 3/4$ and sketch the positive branch in your solution. Explain why an explicit formula obtained by solving the cubic is unnecessary for these conclusions.

Question 8 – Solution

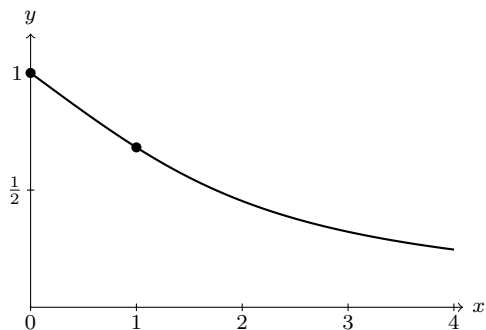
Strategy. Establish a unique branch before differentiating. Monotonicity and inequalities can describe an implicit solution without an explicit cubic formula.

Step 1: A well-defined positive branch. For fixed $x \geq 0$, let $H(y) = y^3 + xy$. It is continuous, starts at $H(0) = 0$, and tends to infinity as $y \rightarrow \infty$. For $y > 0$, $H'(y) = 3y^2 + x > 0$, so the Intermediate Value Theorem and strict increase give exactly one positive root of $H(y) = 1$.

Step 2: Verification. Differentiation yields

$$3y^2y' + y + xy' = 0, \quad \boxed{y' = -\frac{y}{x + 3y^2}}.$$

The denominator is positive on the branch, allowing local differentiable continuation, including near $(0, 1)$. At zero the relation gives $y^3 = 1$, so $y(0) = 1$ and $\boxed{y'(0) = -1/3}$.



The positive branch of $y^3 + xy = 1$; dots at $x = 0$ and $x = 1$.

Step 3: Decrease and limit. Positivity of y and the denominator implies $y' < 0$. Also $y^3 + xy = 1$ gives $0 < y < 1/x$ for $x > 0$, so the squeeze principle gives $\boxed{y(x) \rightarrow 0 \text{ as } x \rightarrow \infty}$.

Step 4: A rigorous bracket. At $x = 1$,

$$(2/3)^3 + 2/3 = 26/27 < 1, \quad (3/4)^3 + 3/4 = 75/64 > 1.$$

Strict increase in y proves $\boxed{2/3 < y(1) < 3/4}$. Root existence, differentiation and inequalities establish these results directly; an explicit cubic formula adds no needed information.

Question 9

A tank initially holds 10 liters of water containing 30 grams of salt. Fresh water enters at 2 liters per minute, and a well-mixed solution leaves at 4 liters per minute. Before the tank empties, the model is

$$V(t) = 10 - 2t, \quad S'(t) = -\frac{4S(t)}{10 - 2t}, \quad S(0) = 30,$$

where S is salt mass in grams. A candidate is

$$S(t) = 30(1 - t/5)^2.$$

Tasks

1. Explain the units and physical meaning of the right-hand side of the salt equation. Verify the candidate wherever that equation is defined.
2. State the largest mathematical solution interval containing 0, and the physical time interval for the stated tank model.
3. Find the concentration $C(t) = S(t)/V(t)$ before emptying and the limits of S and C as the emptying time is approached.
4. Plot salt mass on the physical interval in your solution. Explain why the polynomial formula does not validate continued operation of this model after the tank empties, and distinguish a concentration limit from a concentration assigned to an empty tank.

Question 9 – Solution

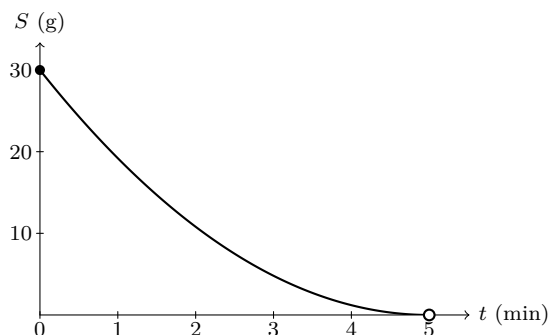
Strategy. Separate algebraic verification from the equation's domain and from the physical assumptions behind it.

Step 1: Units and substitution. The concentration S/V has units g/L; multiplying by the outflow 4 L/min gives salt loss in g/min. Fresh inflow adds no salt. The candidate satisfies

$$S' = -12(1 - t/5), \quad -\frac{4S}{10 - 2t} = -12(1 - t/5) \quad (t \neq 5),$$

and $S(0) = 30$.

Step 2: Two different intervals. The differential equation is undefined at $t = 5$. The largest open mathematical interval containing the initial time is $(-\infty, 5)$. The stated experiment starts at 0 and requires positive volume, so its physical interval is $0 \leq t < 5$ minutes.



Solid: salt mass for $0 \leq t < 5$. Open endpoint: emptying time is excluded.

Step 3: Concentration and endpoint limits. Before emptying,

$$C(t) = 3(1 - t/5) \text{ g/L}, \quad \lim_{t \rightarrow 5^-} S(t) = 0, \quad \lim_{t \rightarrow 5^-} C(t) = 0.$$

At $t = 5$, both S and V have zero limits. The quotient S/V is not defined for the empty tank, even though the pre-emptying concentration has a limit.

Step 4: Why a polynomial is not a license to extrapolate. The polynomial can be evaluated after 5, but $V(t)$ would then be negative and the assumed outflow could not be maintained from this empty tank. The algebraic formula is not sufficient to preserve the model's physical meaning or extend the original ODE across its undefined point.

Question 10

A population model in the original units is

$$\frac{dP}{dt} = \frac{1}{5}P(1 - P/50), \quad P(0) = 10,$$

where t is measured in days and 50 has the same population units as P . Define dimensionless variables

$$\tau = t/5, \quad u(\tau) = P(5\tau)/50.$$

A candidate for a normalized model is

$$u(\tau) = \frac{1}{1 + 4e^{-\tau}}.$$

Tasks

1. Use the chain rule to derive the equation and initial value for u .
2. Verify the candidate in the normalized variables, then express the corresponding $P(t)$ in the original variables.
3. Find when the population first reaches 25, giving both dimensionless time and time in days.
4. Explain why changing units alters numerical slopes and time labels without producing a different physical prediction. Identify what would go wrong if the factor from $t = 5\tau$ were omitted in differentiation.

Question 10 – Solution

Strategy. Transform both the dependent variable and time. The chain-rule factor is essential when comparing numerical rates in different units.

Step 1: Transform the IVP. Since $u(\tau) = P(5\tau)/50$,

$$\frac{du}{d\tau} = \frac{5}{50}P'(5\tau) = \frac{5}{50} \cdot \frac{1}{5}(50u)(1-u) = u(1-u).$$

The initial value is $u(0) = 10/50 = 1/5$. Hence

$$u' = u(1-u), \quad u(0) = 1/5.$$

Step 2: Verify and return to original units. The supplied function satisfies

$$u' = \frac{4e^{-\tau}}{(1+4e^{-\tau})^2} = u(1-u), \quad u(0) = 1/5.$$

Substituting $\tau = t/5$ gives

$$P(t) = \frac{50}{1+4e^{-t/5}}.$$

As a direct check, $P' = 40e^{-t/5}/(1+4e^{-t/5})^2$ agrees with the original right-hand side and $P(0) = 10$.

Step 3: First passage to half capacity. The candidate increases strictly and reaches $P = 25$ when $u = 1/2$. Therefore

$$1+4e^{-\tau} = 2 \implies \tau = \ln 4, \quad t = 5 \ln 4 \text{ days.}$$

Step 4: Same prediction, different labels. One unit of τ is five days, and one unit of u is fifty population units. Omitting the factor 5 in $du/d\tau = (5/50)P'(5\tau)$ would incorrectly give $u' = u(1-u)/5$ and misrepresent the normalized time scale. A change of units must transform rates as well as values.