

Question 1

Consider the direction field for

$$y' = x - y.$$

An *isocline of slope k* is the set of points where the equation assigns slope k . A direction field consists of short line segments with the prescribed slopes; the segments are not themselves whole solution curves.

Tasks

1. Calculate the slopes at $(-1, 1)$, $(0, 1)$, $(1, 1)$, and $(1, -1)$, and describe the orientation of each segment.
2. Find the isoclines of slopes -1 , 0 , and 1 . Determine the regions of positive and negative slope.
3. Draw a direction field on $-2 \leq x \leq 2$, $-2 \leq y \leq 2$ in your solution. Mark the three isoclines and distinguish them from solution curves.
4. A solution passes through $(0, 1)$. Find its tangent line there and decide whether it initially rises or falls as x increases. Does the tangent line itself solve the equation on an interval?

Question 1 – Solution

Strategy. Evaluate $x - y$ at points, then hold its value fixed to find isoclines. Test a proposed curve by comparing its derivative with the assigned slope.

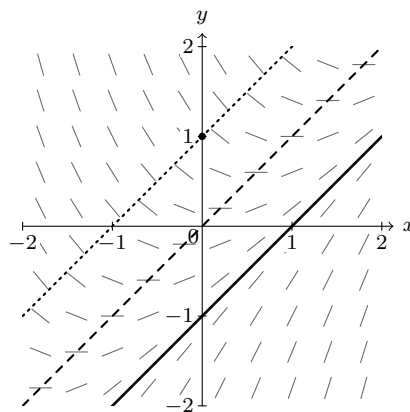
Step 1: Slopes and regions. The four slopes, in the listed order, are

$$\boxed{-2, \quad -1, \quad 0, \quad 2.}$$

Thus the first two segments fall to the right, the third is horizontal, and the fourth rises. The isocline equation is $x - y = k$, or $y = x - k$. Hence

$$k = -1 : y = x + 1, \quad k = 0 : y = x, \quad k = 1 : y = x - 1.$$

Below $y = x$ the slopes are positive; above it they are negative.



Solid: $y = x - 1$; dashed: $y = x$; dotted: $y = x + 1$. Dot: $(0, 1)$.

Step 2: Isoclines are not automatically solutions. Each displayed isocline has geometric slope 1. On $y = x + 1$ and $y = x$, the field instead prescribes -1 and 0 , so these lines are not solutions. On $y = x - 1$, the prescribed slope is 1, so that particular isocline also happens to be a solution.

Step 3: Tangent versus solution. At $(0, 1)$ the slope is -1 , giving $\boxed{y = 1 - x}$ as the tangent line. The solution initially falls. Along this line, however, $x - y = 2x - 1$, which equals the line's derivative -1 only at $x = 0$. Thus the tangent line does not solve the equation on any open interval.

Question 2

Consider the autonomous equation

$$y' = y(2 - y).$$

You may use the following fact: for this smooth right-hand side, two solution curves cannot intersect at a common point unless they coincide on their common interval. Assume the solution with $y(0) = 1$ exists for all $x \geq 0$.

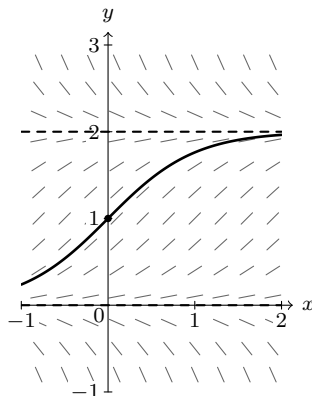
Tasks

1. Explain why the direction field repeats horizontally, and find all horizontal solution lines.
2. Determine the slope sign in each region separated by those lines. Draw the field for $-1 \leq x \leq 2$ and $-1 \leq y \leq 3$ in your solution.
3. Prove that the solution with $y(0) = 1$ remains between 0 and 2 and is increasing for $x \geq 0$.
4. Determine its limit as $x \rightarrow \infty$. Justify why a limiting value strictly below 2 would contradict the differential equation; do not solve the equation explicitly.

Question 2 – Solution

Strategy. Read the signs between horizontal zero-slope lines, then use nonintersection and a bounded-monotone argument.

Step 1: Field structure. The slope depends only on y , so it is the same along every horizontal row. Constant solutions require $y(2 - y) = 0$, giving $y = 0$ and $y = 2$. The sign is negative for $y < 0$, positive for $0 < y < 2$, and negative for $y > 2$.



Solid: solution through $(0, 1)$. Dashed: the equilibrium lines $y = 0, 2$.

Step 2: Confinement and monotonicity. Starting at 1, a continuous solution could leave $(0, 2)$ only by meeting one of the equilibrium solutions. The stated nonintersection fact excludes this. Therefore

$$0 < y(x) < 2, \quad y'(x) > 0 \quad (x \geq 0).$$

Since $y(0) = 1$, the function is bounded below by 1 on this half-line.

Step 3: Limit without an explicit formula. An increasing function bounded above by 2 has a limit L with $1 \leq L \leq 2$. If $L < 2$, continuity of $g(y) = y(2 - y)$ gives $g(L) > 0$ and, eventually, $y'(x) \geq g(L)/2 =: c > 0$. Integrating from a sufficiently large X yields

$$y(x) \geq y(X) + c(x - X),$$

contradicting the upper bound 2. Thus $\lim_{x \rightarrow \infty} y(x) = 2$. The limit is approached from below, without touching the equilibrium at any finite x .

Question 3

For the nonautonomous equation

$$y' = x + y,$$

a student claims that the zero-slope isocline is a solution because “every field segment on it is horizontal.” A candidate through the origin is supplied:

$$v(x) = e^x - x - 1.$$

Tasks

1. Find the zero-slope isocline and the regions where solutions rise or fall. Test the student’s claim.
2. Verify that v solves the equation and satisfies $v(0) = 0$.
3. Determine whether the horizontal tangent at the origin is a maximum, a minimum, or neither. Explain how v crosses the zero-slope isocline.
4. Draw the field, the isocline, and the verified solution near the origin in your solution. Explain why crossing an isocline does not contradict uniqueness of solution curves.

Question 3 – Solution

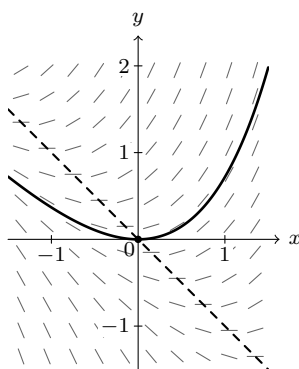
Strategy. Distinguish a line made of zero-slope locations from a curve whose own tangent agrees with the field everywhere.

Step 1: Zero slopes. The zero-slope isocline is $y = -x$. Slopes are positive above it and negative below it. The line itself has derivative -1 , while the equation assigns 0 along it, so it is not a solution.

Step 2: Verify the candidate. We have

$$v' = e^x - 1 = x + (e^x - x - 1) = x + v, \quad v(0) = 0.$$

Thus v is a solution on all of $\mathbb{R}$.



Solid: $v = e^x - x - 1$. Dashed: the zero-slope isocline $y = -x$.

Step 3: Classify the contact. Since $e^x - 1 < 0$ for $x < 0$ and $e^x - 1 > 0$ for $x > 0$, v decreases before 0 and increases after 0 . Therefore $(0, 0)$ is a strict minimum. Also $v'' = e^x > 0$ everywhere.

Step 4: Crossing the reference line. The signed vertical difference from $y = -x$ is

$$v(x) - (-x) = e^x - 1.$$

It changes from negative to positive at 0 , so the solution crosses the isocline from below to above. Nonintersection of distinct solutions is irrelevant: the isocline is not a solution curve.

Question 4

An unknown direction field is known to come from an affine law

$$y' = ax + by + c,$$

with fixed real constants a, b, c . Three exact slope measurements are

(x, y)	$(-1, 0)$	$(0, 1)$	$(1, 1)$
slope	-1	0	2

A fourth report claims that the slope at $(2, 0)$ is 4.

Tasks

1. Recover a, b, c and show that the first three measurements determine them uniquely.
2. Decide whether the fourth report is consistent with the model.
3. Find the zero-slope isocline. Then determine every straight-line solution $y = mx + d$ of the recovered equation.
4. Draw the recovered field, its zero-slope isocline, and its straight-line solution in your solution. Explain why the affine-model assumption matters when inferring a whole field from three points.

Question 4 – Solution

Strategy. Recover the coefficients from independent data, then compare the geometric slope of a line with the field along it.

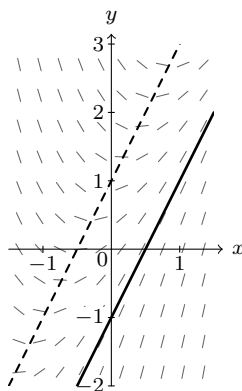
Step 1: Recover the law. The measurements give

$$-a + c = -1, \quad b + c = 0, \quad a + b + c = 2.$$

Subtracting the second equation from the third gives $a = 2$. The first then gives $c = 1$, and the second gives $b = -1$. These forced values prove uniqueness within the affine class:

$$y' = 2x - y + 1.$$

At $(2, 0)$ the predicted slope is 5, so the reported value 4 is inconsistent.



Solid: solution $y = 2x - 1$. Dashed: zero-slope isocline $y = 2x + 1$.

Step 2: Reference line and solution line. Zero slopes occur on $y = 2x + 1$, which is not a solution: its derivative is 2, not 0. Substituting $y = mx + d$ into the equation requires

$$m = (2 - m)x + 1 - d$$

for every x in an interval. Hence $m = 2$ and $d = -1$. The only straight-line solution is $y = 2x - 1$, verified by $2x - (2x - 1) + 1 = 2$.

Step 3: Limits of inference. Three data points determine the three affine coefficients here. Without the affine assumption, many nonlinear functions could agree at those points while assigning different slopes elsewhere. Finite measurements alone do not determine an unrestricted field.

Question 5

Consider the equation

$$y' = \frac{x}{y}, \quad y \neq 0,$$

and a solution through $(0, 1)$.

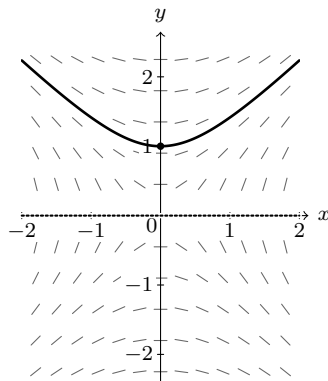
Tasks

1. Determine the slope sign in each open quadrant. Find every point in the domain with a horizontal field segment.
2. Explain why no segment should be placed at $(1, 0)$, even though nearby segments become steep.
3. Along a solution, differentiate $y^2 - x^2$. Use the initial point to identify an explicit solution branch and verify it directly. State its largest open interval.
4. Draw the field and the selected branch in your solution, marking the excluded line $y = 0$. Explain why that excluded line does not prevent this particular solution from existing for every real x .

Question 5 – Solution

Strategy. Respect the equation's domain before drawing. Use a constant quantity along a solution to identify the curve through the given point.

Step 1: Signs and missing points. Slopes are positive in Quadrants I and III, and negative in II and IV. Horizontal segments occur exactly where $x = 0$ and $y \neq 0$. The expression x/y is undefined on all of $y = 0$; a vertical mark at $(1, 0)$ would falsely assign a direction to a point outside the domain.



Solid: $y = \sqrt{1 + x^2}$. Dotted horizontal line: excluded $y = 0$.

Step 2: Identify and verify the branch. Wherever a solution is defined,

$$\frac{d}{dx}(y^2 - x^2) = 2yy' - 2x = 2y\frac{x}{y} - 2x = 0.$$

Thus $y^2 - x^2$ is constant on its interval. At $(0, 1)$ it equals 1. Continuity and the positive initial value select

$$y = \sqrt{1 + x^2}, \quad -\infty < x < \infty.$$

Indeed $y' = x/\sqrt{1 + x^2} = x/y$ and $y(0) = 1$.

Step 3: Interval and geometry. This branch is smooth and satisfies $y \geq 1$ for every real x , so it never reaches the excluded axis. The entire real line is therefore its largest open interval. It decreases for $x < 0$ and increases for $x > 0$, with a minimum at $(0, 1)$, matching the field.

Question 6

Compare two equations on the same coordinate scale:

$$(A) \quad y' = y, \quad (B) \quad y' = 3y.$$

Both are assigned the initial value $y(0) = 1$. Candidate solutions are supplied:

$$u(x) = e^x, \quad v(x) = e^{3x}.$$

Tasks

1. Determine the zero-slope locations and slope signs for both fields. Are these facts enough to make the two fields identical?
2. At height $y = 1$, compute each slope and its angle with the positive x -axis. If the two fields use equal-length segments, explain how steepness should be represented.
3. Verify the two candidates and calculate when each first reaches $y = 2$ for $x > 0$.
4. Draw the two fields and their respective verified curves on identical axes in your solution. State precisely which geometric information is lost by recording only slope signs.

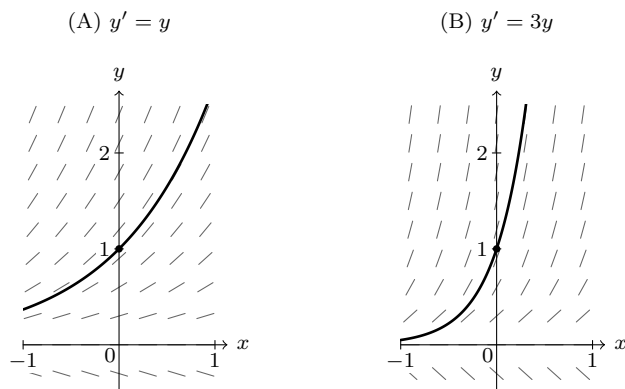
Question 6 – Solution

Strategy. Separate slope sign from slope magnitude. Equal-length field segments encode magnitude through their angle, not their length.

Step 1: Shared and different information. Both fields have horizontal segments on $y = 0$, positive slopes above it, and negative slopes below it. But at $y = 1$ their slopes are 1 and 3, so the fields are different. The corresponding angles are

$$\theta_A = \arctan(1) = 45^\circ, \quad \theta_B = \arctan(3) \approx 71.6^\circ.$$

A segment of total length ℓ and slope m can use the displacement $\ell(1, m)/\sqrt{1 + m^2}$. This preserves its slope while controlling its length.



Solid curves: e^x on the left and e^{3x} on the right; identical axis scales.

Step 2: Verify and compare growth. Directly,

$$u' = e^x = u, \quad v' = 3e^{3x} = 3v, \quad u(0) = v(0) = 1.$$

Both increase strictly. Solving $e^x = 2$ and $e^{3x} = 2$ gives their first hitting times

$$x_A = \ln 2, \quad x_B = \frac{\ln 2}{3}.$$

Thus (B) reaches the same height in one-third the time.

Step 3: What signs omit. Signs indicate rising, falling, or horizontal tangents. They do not specify tangent angles, rates of change, or travel times between heights. Identical sign regions and the same equilibrium line do not determine the same solution curves.

Question 7

For the equation

$$y' = x^2 - y,$$

consider the supplied candidate through the origin

$$w(x) = x^2 - 2x + 2 - 2e^{-x}.$$

You may use the calculus inequality $e^t > 1 + t$ for $t \neq 0$.

Tasks

1. Find the zero-slope isocline and describe the slope signs above and below it.
2. Verify that w solves the equation and that $w(0) = w'(0) = 0$.
3. Decide whether w has a local extremum at 0. Determine its concavity on either side of 0 and classify the horizontal tangent there.
4. Draw the field, the isocline, and w near the origin in your solution. Explain why observing a horizontal field segment is insufficient to conclude that a solution has a maximum or minimum.

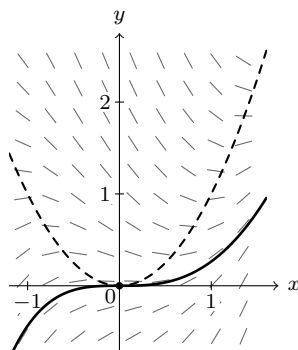
Question 7 – Solution

Strategy. A zero first derivative identifies a stationary point, not its type. Inspect nearby derivative signs and concavity.

Step 1: Isocline and verification. Zero slopes occur on $y = x^2$. Slopes are positive below this parabola and negative above it. For the candidate,

$$w' = 2x - 2 + 2e^{-x} = x^2 - w, \quad w(0) = w'(0) = 0.$$

It is therefore a solution on $\mathbb{R}$ with a horizontal tangent at the origin.



Solid: $w = x^2 - 2x + 2 - 2e^{-x}$. Dashed: $y = x^2$.

Step 2: No local extremum. For $x \neq 0$, the supplied inequality gives $e^{-x} > 1 - x$, so

$$w' = 2(x - 1 + e^{-x}) > 0.$$

The derivative is positive on both sides of 0. Thus w increases through the origin and has neither a local maximum nor a local minimum there. Moreover, $x^2 - w = w' > 0$ for $x \neq 0$, so the curve lies below its zero-slope isocline on both sides.

Step 3: Concavity. Since $w'' = 2 - 2e^{-x}$, the curve is concave down for $x < 0$ and concave up for $x > 0$. Hence

$(0, 0)$ is an inflection point with a horizontal tangent.

The field gives the tangent slope at a point; the surrounding behavior is needed to classify that point.

Question 8

Consider

$$y' = xy.$$

For any solution y on an interval I , define the reflected functions

$$g(x) = y(-x) \quad (x \in -I), \quad h(x) = -y(x) \quad (x \in I),$$

where $-I = \{-t : t \in I\}$. Also consider the candidate $p(x) = e^{x^2/2}$.

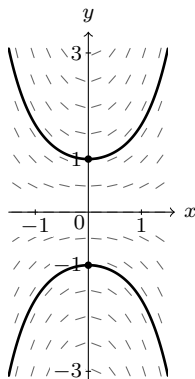
Tasks

1. Determine the zero-slope locations and the slope signs in the four quadrants.
2. Verify by differentiation that both reflected functions satisfy the same equation on their stated intervals. Explain why reflecting across the vertical axis requires a sign change in the derivative.
3. Verify p , find its minimum, and describe the corresponding extremum of $-p$.
4. Draw the field with p and $-p$ in your solution. Explain how the graph exhibits both reflection symmetries without treating a direction segment as an arrow giving a unique direction of travel.

Question 8 – Solution

Strategy. Check symmetry by transforming the differential equation itself, using the chain rule for reflection in the independent variable.

Step 1: Slopes. Since $xy = 0$ on both coordinate axes, all field segments there are horizontal. Slopes are positive in Quadrants I and III and negative in II and IV. The horizontal axis is an equilibrium solution; the vertical axis is not a function $y(x)$.



Solid: $p = e^{x^2/2}$ and $-p$. The field and curves respect both reflections.

Step 2: Transform the derivatives. For $x \in -I$, the chain rule gives

$$g'(x) = -y'(-x) = -[(-x)y(-x)] = xg(x).$$

The minus sign comes from differentiating $-x$. For $x \in I$,

$$h'(x) = -y'(x) = -xy(x) = xh(x).$$

Thus each reflection takes solutions to solutions on the stated reflected or original interval.

Step 3: Verify the displayed curves. We have $p' = xe^{x^2/2} = xp$ and $p(0) = 1$. Because $x^2/2 \geq 0$,

$\min_{\mathbb{R}} p = 1 \text{ at } x = 0, \quad \max_{\mathbb{R}}(-p) = -1 \text{ at } x = 0.$
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The curve p is even, and $-p$ is its reflection across the horizontal axis. Field segments indicate tangent lines. When following a graph in the direction of increasing x , the derivative sign specifies whether y rises or falls; the drawn segments themselves need no arrowheads.

Question 9

A plotting program samples slopes only at the five columns

$$x = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$$

and at any chosen collection of y -values. Compare the two smooth equations

$$(A) \quad y' = 0, \quad (B) \quad y' = \sin(2\pi x).$$

Both have initial value $y(0) = 0$.

Tasks

1. Prove that the sampled direction fields are identical at every plotted point.
2. Add the column $x = \frac{1}{4}$. Calculate the slopes assigned by each law there and explain why it reveals the difference.
3. Verify the candidates $u(x) = 0$ and $v(x) = [1 - \cos(2\pi x)]/(2\pi)$ for the respective IVPs. Compare their values at $x = \frac{1}{2}$.
4. In your solution, display the shared coarse field and a more finely sampled field for (B), together with the corresponding curves. Explain why a finite sampled field cannot determine an arbitrary smooth differential equation uniquely.

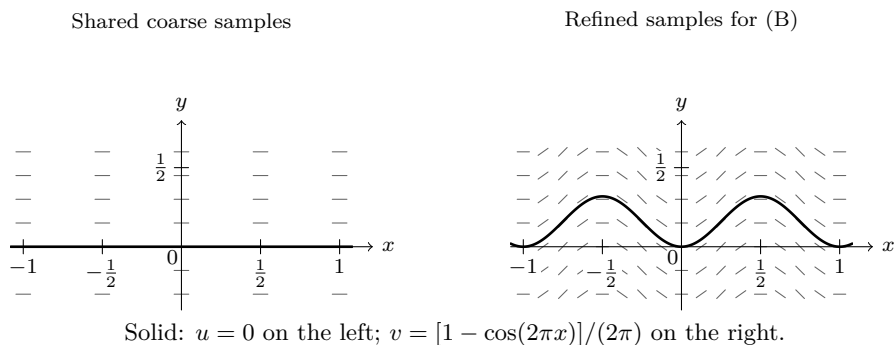
Question 9 – Solution

Strategy. Evaluate the law at the actual sampling locations before trusting a plot. A grid can systematically miss variation between columns.

Step 1: Hidden variation. Every sampled column has $x = j/2$ for an integer j . Therefore $\sin(2\pi x) = \sin(j\pi) = 0$. Both programs draw only horizontal segments there, regardless of y . At the added column,

$$f_A(\tfrac{1}{4}, y) = 0, \quad f_B(\tfrac{1}{4}, y) = 1,$$

so the fields visibly differ.



Step 2: Distinct solutions. Direct differentiation gives $u' = 0$ and

$$v'(x) = \frac{2\pi \sin(2\pi x)}{2\pi} = \sin(2\pi x), \quad u(0) = v(0) = 0.$$

At the half-unit point, $u(\frac{1}{2}) = 0, \quad v(\frac{1}{2}) = 1/\pi$. Equal sampled slopes do not imply equal solution values between or even at later grid columns.

Step 3: What a refined grid proves. The extra column disproves equality of these two laws. It does not make finite sampling sufficient for an unrestricted smooth law. For any finite set of sampled columns x_j , the nonzero smooth function $\prod_j (x - x_j)^2$ vanishes at every one of them. Adding it to a right-hand side preserves those sampled slopes while changing the field elsewhere.

Question 10

A scaled temperature model uses time x and temperature y :

$$y' = T(x) - y, \quad T(x) = 2 + \sin x.$$

The prescribed ambient temperature T varies in time. A periodic response is proposed:

$$r(x) = 2 + \frac{\sin x - \cos x}{2}.$$

Tasks

1. Interpret the slope signs above and below $y = T(x)$. Compute the slopes at $(0, 1)$, $(0, 3)$, and $(\pi/2, 2)$.
2. Test whether the ambient-temperature curve itself solves the equation. Verify that r does solve it.
3. On $0 \leq x \leq 2\pi$, find the maximum and minimum of r , including their times. Compare the time of its maximum with the time of the ambient maximum.
4. Draw the field, the ambient curve, and the response in your solution. Explain why a response extremum occurs where the response meets the zero-slope curve, although that curve is not itself a solution.

Question 10 – Solution

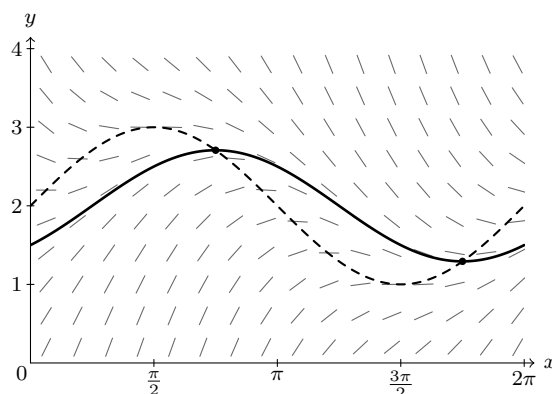
Strategy. Interpret the moving zero-slope curve as a temperature comparison, then verify the response and locate its horizontal tangents.

Step 1: Heating and cooling. Below $y = T(x)$, $y' > 0$ and the object warms; above it, $y' < 0$ and it cools. At the three listed points, the slopes are $\boxed{1, -1, 1}$.

Step 2: Check both curves. Along $y = T(x)$ the right-hand side is zero, but $T'(x) = \cos x$ is not identically zero on any interval. Thus T is not a solution. By contrast,

$$r' = \frac{\cos x + \sin x}{2} = 2 + \sin x - r,$$

so r is a solution for all real x .



Solid: response r . Dashed: ambient temperature T . Dots: response extrema.

Step 3: Extrema and delay. Rewrite

$$r(x) = 2 + \frac{1}{\sqrt{2}} \sin(x - \pi/4).$$

On the stated interval, its extreme values are

$$\boxed{r_{\max} = 2 + \frac{1}{\sqrt{2}} \text{ at } x = \frac{3\pi}{4}, \quad r_{\min} = 2 - \frac{1}{\sqrt{2}} \text{ at } x = \frac{7\pi}{4}.$$

The endpoint values both equal $3/2$ and lie between these extremes. The ambient maximum occurs at $x = \pi/2$, so the response maximum follows it by $\boxed{\pi/4}$ time units.

Step 4: Field interpretation. At a response extremum, $r' = T - r = 0$, so the response meets the zero-slope curve. Elsewhere it need not equal T . A locus of zero assigned slopes can move with time without being a solution curve itself.