

Question 1

Consider the following equations. In (A)–(D), y is a function of x ; in (E), u is a function of the independent variables x and t .

(A) $x^2 y'' + \sin(x) y' - 3y = e^x$,

(B) $y'' + y y' = 0$,

(C) $(y')^2 + y = x$,

(D) $y'' + \sin(y) = 0$,

(E) $u_t = 4u_{xx} + x u$.

Tasks

1. Identify each equation as an ordinary or partial differential equation, and state its order.
2. Decide whether each equation is linear in its unknown function. For every nonlinear equation, identify the precise obstruction to linearity.
3. Among the linear equations, identify which are homogeneous after all unknown-function terms are moved to the left.
4. Explain why the factor x^2 in (A) and the factor x in (E) do not cause nonlinearity.

Question 1 – Solution

Strategy. Count the highest derivative order, then examine how the unknown function and its derivatives occur. A linear equation has coefficients depending only on independent variables.

Step 1: Classification.

Equation	Type	Order	Linearity
(A)	ODE	2	Linear
(B)	ODE	2	Nonlinear: product $y y'$
(C)	ODE	1	Nonlinear: square $(y')^2$
(D)	ODE	2	Nonlinear: $\sin(y)$
(E)	PDE	2	Linear

In (E), u_t has order one and u_{xx} has order two; the highest order is therefore two. In (C), squaring a first derivative does not make it a second derivative.

Step 2: Homogeneity. The two linear equations can be written as

$$x^2 y'' + \sin(x) y' - 3y = e^x, \quad u_t - 4u_{xx} - xu = 0.$$

Thus (A) is **nonhomogeneous**, and (E) is **homogeneous**. Here “homogeneous” is being used in the linear-equation sense.

Step 3: Coefficients versus unknowns. The factors x^2 , $\sin(x)$, and x are specified functions of independent variables, so they are allowed coefficients. By contrast, y multiplying y' in (B) is part of the unknown solution.

Check. Order and linearity are different properties: (C) is first order but nonlinear, whereas (A) is second order but linear. The vanishing of x^2 at 0 affects normalization of (A), not this linearity classification.

Question 2

For the initial value problem

$$y' + 2y = 6, \quad y(0) = 1,$$

three candidate functions are proposed:

$$f(x) = 3 - 2e^{-2x}, \quad g(x) = 3 - 2e^{2x}, \quad h(x) = 3.$$

Tasks

1. For each candidate, calculate its residual $R[v] = v' + 2v - 6$ and its value at $x = 0$.
2. Determine which candidates solve the differential equation and which solve the initial value problem. State an interval for each accepted solution.
3. Within the family $y = 3 + Ce^{-2x}$, determine the parameter selected by the initial condition.
4. Explain why matching one initial value, or satisfying an equation at a single point, does not verify a solution of an initial value problem.

Question 2 – Solution

Strategy. Verify two separate requirements: the differential equation must hold throughout an interval, and the initial value must hold at the specified point.

Step 1: Residuals. Direct differentiation gives

$$\begin{aligned}f' &= 4e^{-2x}, & R[f] &= 4e^{-2x} + 6 - 4e^{-2x} - 6 = 0, \\g' &= -4e^{2x}, & R[g] &= -4e^{2x} + 6 - 4e^{2x} - 6 = -8e^{2x}, \\h' &= 0, & R[h] &= 0 + 6 - 6 = 0.\end{aligned}$$

Also $f(0) = 1$, $g(0) = 1$, and $h(0) = 3$.

Step 2: Decisions. Since $e^{2x} > 0$, the residual for g never vanishes. Hence

f and h solve the differential equation on $\mathbb{R}$,
 f alone among these candidates solves the IVP on $\mathbb{R}$.

The function g passes the initial-value test but fails the equation; h does the reverse.

Step 3: Parameter selection. For $y = 3 + Ce^{-2x}$,

$$y' = -2Ce^{-2x}, \quad y' + 2y = 6.$$

Every real C gives an equation solution on $\mathbb{R}$. The initial condition requires $3 + C = 1$, so $C = -2$, recovering f .

Step 4: Logical distinction. An initial value is one pointwise constraint. A differential equation is an identity along the candidate function on an interval. For example, $v(x) = 3 + x^2$ satisfies $v' + 2v = 6$ at $x = 0$, but its residual $2x + 2x^2$ is not identically zero on any open interval. Neither pointwise test replaces interval verification.

Question 3

An implicit relation and an initial value are given by

$$x^2 + y^2 = 9, \quad y(0) = 3.$$

The proposed differential equation is

$$y' = -\frac{x}{y}.$$

Tasks

1. Identify the explicit branch selected by the initial value and verify the differential equation.
2. Find the largest open interval containing 0 on which that branch is a classical solution of the displayed equation.
3. Explain why the entire circle is not one solution function $y(x)$, and why adding the endpoints of the upper semicircle does not extend the solution interval.
4. If the initial value is changed to $y(0) = -3$, identify the corresponding branch and its interval. Sketch both branches in your solution.

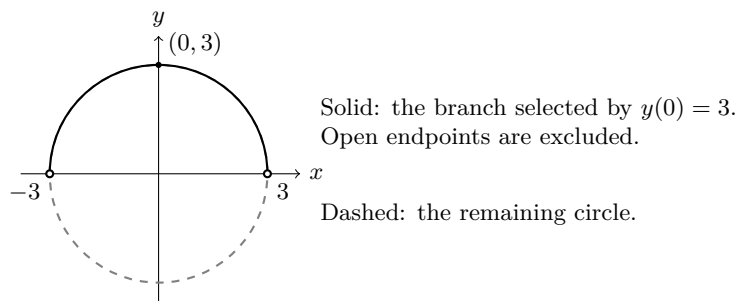
Question 3 – Solution

Strategy. An implicit curve can contain multiple function branches. Select the branch using the initial value, then check both differentiability and the equation's denominator.

Step 1: Select and verify. Solving for y gives $y = \pm\sqrt{9-x^2}$. Since $y(0) = 3$, choose

$$y(x) = \sqrt{9-x^2}, \quad -3 < x < 3.$$

On this interval, $y' = -x/\sqrt{9-x^2} = -x/y$, and $y(0) = 3$.



Step 2: Interval and endpoints. At $x = \pm 3$, the branch has $y = 0$, so $-x/y$ is undefined and the derivative is not finite. For $|x| > 3$, this branch is not real. Thus the largest open interval containing 0 is exactly $(-3, 3)$.

Step 3: Relation versus function. For each $-3 < x < 3$, the circle has two distinct y -values. The whole circle fails the single-valued requirement for a function $y(x)$. Even the undivided equation $yy' = -x$ cannot hold with a finite derivative at $(\pm 3, 0)$: it would require $0 = \mp 3$.

Step 4: Other initial value. For $y(0) = -3$,

$$y(x) = -\sqrt{9-x^2}, \quad -3 < x < 3.$$

Its derivative is $x/\sqrt{9-x^2} = -x/y$, verifying the lower branch separately.

Question 4

Consider

$$y' = y(1 - y).$$

A student divides by $y(1 - y)$ and claims that

$$\frac{y'}{y(1 - y)} = 1$$

is an equivalent equation for every solution. The following functions are available for testing:

$$y_0(x) = 0, \quad y_1(x) = 1, \quad p(x) = \frac{e^x}{1 + e^x}.$$

Tasks

1. Find all constant solutions of the original equation.
2. Verify p directly and state its domain and range.
3. Explain exactly when the division is valid and which listed solutions it excludes.
4. Match the initial values $y(0) = 0$, $y(0) = 1$, and $y(0) = \frac{1}{2}$ to the listed candidates. Explain why this matching alone is not a proof of uniqueness among all possible solutions.

Question 4 – Solution

Strategy. Check the original equation before transforming it. Division by an expression involving the unknown can discard legitimate solutions.

Step 1: Constant solutions. If $y(x) = k$, then $y' = 0$, so

$$0 = k(1 - k) \iff \boxed{k = 0 \text{ or } k = 1.}$$

Both constant functions solve the equation on $\mathbb{R}$; these are its equilibrium solutions.

Step 2: Nonconstant verification. The quotient rule gives

$$p'(x) = \frac{e^x(1 + e^x) - e^{2x}}{(1 + e^x)^2} = \frac{e^x}{(1 + e^x)^2}.$$

Also

$$p(1 - p) = \frac{e^x}{1 + e^x} \frac{1}{1 + e^x} = p'.$$

The denominator is positive for every real x . Because e^x ranges over $(0, \infty)$,

$$\boxed{\text{domain}(p) = \mathbb{R}, \quad \text{range}(p) = (0, 1).}$$

Step 3: Restricted equivalence. Division is valid on an interval only if $y(x) \neq 0$ and $y(x) \neq 1$ everywhere on that interval. Thus it is valid for p , but is undefined for both equilibrium solutions. The equations are equivalent under the nonvanishing restriction, not for every original solution.

Step 4: Initial data and scope. Direct substitution yields

$$\boxed{y_0(0) = 0, \quad y_1(0) = 1, \quad p(0) = \frac{1}{2}.}$$

Each is therefore a verified solution of the corresponding IVP. Testing three candidates does not exclude every other function: existence of a matching candidate and uniqueness of an IVP are logically different claims.

Question 5

Consider the initial value problem

$$y' = 2\sqrt{y}, \quad y(0) = 0,$$

where the square root is the nonnegative real root. For each real parameter $a \geq 0$, define on $\mathbb{R}$

$$y_a(x) = \begin{cases} 0, & x \leq a, \\ (x - a)^2, & x > a. \end{cases}$$

Tasks

1. Prove that y_a is continuously differentiable at its joining point $x = a$.
2. Verify the differential equation everywhere, including at $x = a$, and verify the initial value.
3. Show that different parameters give distinct solutions of the same IVP. Identify an additional solution that never leaves 0.
4. Sketch y_0 , y_1 , and y_2 in your solution and explain what this example shows about uniqueness. Does it mean that every first-order IVP is nonunique?

Question 5 – Solution

Strategy. Check the joining point directly; solving on the two sides alone is insufficient. Then compare solutions with the same initial value.

Step 1: Smooth joining. Both pieces tend to 0 as $x \rightarrow a$. At a , the difference quotient is 0 for $h < 0$ and $h^2/h = h$ for $h > 0$. Hence $y'_a(a) = 0$ and

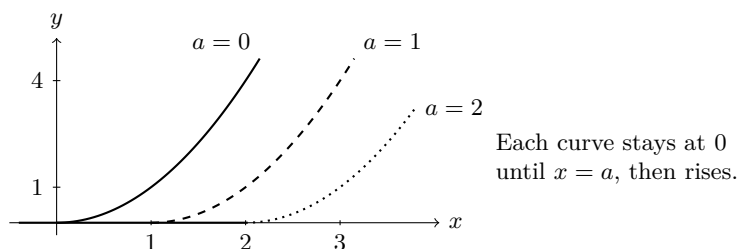
$$y'_a(x) = \begin{cases} 0, & x \leq a, \\ 2(x - a), & x > a. \end{cases}$$

This derivative is continuous at a , so y_a is C^1 on $\mathbb{R}$.

Step 2: Equation and initial value. For $x \leq a$, both y'_a and $2\sqrt{y_a}$ are 0. For $x > a$, $x - a > 0$, so

$$2\sqrt{(x - a)^2} = 2|x - a| = 2(x - a) = y'_a.$$

Since $a \geq 0$, $y_a(0) = 0$. Thus every y_a solves the IVP on $\mathbb{R}$.



Step 3: Distinctness. If $0 \leq a < b$, choose $a < x < b$. Then $y_a(x) > 0$ but $y_b(x) = 0$. Also $y \equiv 0$ is a solution distinct from every finite-parameter y_a .

Step 4: Meaning. This IVP has infinitely many solutions: the initial value does not determine when a solution leaves 0. It disproves automatic uniqueness for all first-order IVPs; it does not assert nonuniqueness for every IVP. No general uniqueness theorem is needed for this counterexample.

Question 6

A one-parameter family of curves is

$$y(x) = (x - C)^2, \quad C \in \mathbb{R}.$$

Tasks

1. Differentiate the family and eliminate C to obtain a first-order differential equation involving only y and y' .
2. Verify that every family member satisfies the resulting equation on $\mathbb{R}$.
3. Find all family members satisfying $y(0) = 1$, and compare their slopes at 0.
4. Test $y \equiv 0$ in your equation. Is it a member of the displayed family on a nonempty open interval? Explain why finding one family of solutions need not describe every solution.

Question 6 – Solution

Strategy. Use the derivative to remove the constant, then investigate what information was lost and whether new solutions are admitted.

Step 1: Eliminate the parameter. Differentiating yields $y' = 2(x - C)$. Squaring and using $y = (x - C)^2$ gives

$$(y')^2 = 4y.$$

This is a first-order nonlinear ODE. The exponent on y' changes linearity, not derivative order.

Step 2: Direct verification. For any real C ,

$$(y')^2 = [2(x - C)]^2 = 4(x - C)^2 = 4y.$$

The family members are polynomials and therefore satisfy the equation on all of $\mathbb{R}$, including $x = C$.

Step 3: Apply the initial value. At $x = 0$ we require $C^2 = 1$, so $C = \pm 1$. The two selected family members are

$$y = (x - 1)^2 \quad \text{and} \quad y = (x + 1)^2.$$

Their slopes at 0 are -2 and 2 , respectively. The equation itself permits either sign because it specifies $(y')^2$, not y' .

Step 4: A solution outside the family. The function $y \equiv 0$ satisfies $(y')^2 = 4y$ identically. If $(x - C)^2$ were identically 0 on a nonempty open interval, every x in that interval would have to equal one fixed number C , which is impossible. Thus

$$y \equiv 0 \text{ is a solution not contained in the displayed family.}$$

Elimination proves that family members solve the equation; it does not prove the converse. In particular, a parameterized family cannot be called a complete description merely because substitution succeeds.

Question 7

You are given that every real solution of $y'' + y = 0$ on $\mathbb{R}$ has the form

$$y(x) = A \cos x + B \sin x.$$

Consider three sets of conditions:

$$(I) \quad y(0) = 2, \quad y'(0) = -1,$$

$$(II) \quad y(0) = 2, \quad y(\pi) = -2,$$

$$(III) \quad y(0) = 2, \quad y(\pi) = 0.$$

Tasks

1. Verify the given family by differentiation.
2. Identify each problem as an initial value problem or a boundary value problem.
3. Determine whether each has exactly one solution, infinitely many solutions, or no solution. Give the corresponding functions when they exist.
4. Explain why the statement “two conditions determine a unique solution of a second-order equation” needs additional qualifications.

Question 7 – Solution

Strategy. Translate each condition into an equation for A and B . Conditions may be independent, redundant, or inconsistent.

Step 1: Verify the family. We have

$$y' = -A \sin x + B \cos x, \quad y'' = -A \cos x - B \sin x = -y.$$

Thus every family member solves $y'' + y = 0$ on $\mathbb{R}$.

Step 2: Initial conditions. In (I), both values are prescribed at $x = 0$, so this is an IVP. Since $y(0) = A$ and $y'(0) = B$,

$$A = 2, \quad B = -1, \quad \boxed{y = 2 \cos x - \sin x.}$$

The supplied completeness of the family makes this the unique solution.

Step 3: Boundary conditions. In (II) and (III), values are prescribed at two different points, so these are BVPs. In both, $y(0) = 2$ gives $A = 2$. But

$$y(\pi) = A \cos \pi + B \sin \pi = -A = -2,$$

independently of B . Consequently,

$$\boxed{\text{(II): } y = 2 \cos x + B \sin x, \quad B \in \mathbb{R}}$$

has infinitely many solutions. In (III), the demand $y(\pi) = 0$ contradicts $y(\pi) = -2$, so $\boxed{\text{(III) has no solution.}}$

Step 4: Why counting fails. Two written conditions need not provide two independent, consistent constraints. In (II) the second repeats information already forced by the first; in (III) it conflicts with that information. Their number alone cannot establish uniqueness or even existence.

Question 8

For a real parameter a , define

$$z_a(x) = \begin{cases} x^2, & x \leq 0, \\ ax^2, & x > 0. \end{cases}$$

Compare the equations

$$(A) \quad xy' = 2y, \quad (B) \quad y'' = 2.$$

A classical solution must have all derivatives appearing in its equation at every point of its open interval of definition.

Tasks

1. Determine for which a the function z_a is C^1 on $\mathbb{R}$, and for which a its second derivative exists at 0.
2. Determine for which a it solves (A) on $\mathbb{R}$, checking $x = 0$ explicitly.
3. Determine for which a it solves (B) on $\mathbb{R}$.
4. Explain why replacing (A) by $y' = 2y/x$ can change the permitted solution intervals, even though the two equations agree whenever $x \neq 0$.

Question 8 – Solution

Strategy. Compute derivatives at the joining point using limits, then test each equation in its original form.

Step 1: First derivative. The pieces meet at $z_a(0) = 0$. The difference quotient at 0 is h for $h < 0$ and ah for $h > 0$, so $z'_a(0) = 0$ for every real a . Therefore

$$z'_a(x) = \begin{cases} 2x, & x < 0, \\ 0, & x = 0, \\ 2ax, & x > 0, \end{cases}$$

which is continuous at 0. Hence $z_a \in C^1(\mathbb{R})$ for every $a \in \mathbb{R}$.

Step 2: Second derivative at the join. The left-hand limit of $[z'_a(h) - z'_a(0)]/h$ is 2, and the right-hand limit is $2a$. Thus

$$z''_a(0) \text{ exists exactly when } a = 1.$$

Step 3: Check equation (A). For $x < 0$, $xz'_a = x(2x) = 2z_a$. For $x > 0$, $xz'_a = x(2ax) = 2z_a$. At $x = 0$, both sides are 0. Consequently all real a give solutions of (A) on $\mathbb{R}$.

Step 4: Check equation (B). On $x > 0$, $z''_a = 2a$, so (B) requires $a = 1$. For that value $z_a = x^2$ globally and $z''_a = 2$, including at 0. Thus $a = 1$ is necessary and sufficient for (B).

Step 5: Domain of the equation. The quotient $2y/x$ is undefined at $x = 0$, even when $y(0) = 0$. The normalized equation therefore cannot have a solution interval containing 0. It agrees with (A) on intervals contained in $(-\infty, 0)$ or $(0, \infty)$, but division removes the original equation's valid point $x = 0$.

Question 9

For a function $u(x, t)$, consider the partial differential equation and initial profile

$$u_t = 4u_{xx}, \quad u(x, 0) = 3 \cos(2x), \quad x \in \mathbb{R}.$$

Search only within the candidate family

$$u(x, t) = Ae^{-\lambda t} \cos(kx), \quad A \neq 0, \quad k > 0,$$

where A , λ , and k are real constants.

Tasks

1. State the independent and dependent variables, the order of the PDE, and whether it is linear and homogeneous.
2. Calculate u_t and u_{xx} and determine exactly which relation among the parameters makes the family satisfy the PDE everywhere.
3. Use the initial profile to determine A , k , and λ , and verify the resulting function.
4. Explain why a condition specifying $u(x, 0)$ supplies more information than one scalar condition such as $u(0, 0) = 3$. Distinguish uniqueness within the proposed family from uniqueness among all PDE solutions.

Question 9 – Solution

Strategy. Treat the other independent variable as constant in each partial derivative. Verify the PDE before imposing the entire initial profile.

Step 1: Definitions. The independent variables are x, t and the dependent variable is u . Written as $u_t - 4u_{xx} = 0$, the equation is second order, linear, and homogeneous.

Step 2: Parameter relation. Direct differentiation gives

$$u_t = -\lambda A e^{-\lambda t} \cos(kx), \quad u_{xx} = -k^2 A e^{-\lambda t} \cos(kx).$$

Thus the residual is

$$u_t - 4u_{xx} = (4k^2 - \lambda) A e^{-\lambda t} \cos(kx).$$

At $x = 0$ the cosine equals 1, while $A e^{-\lambda t} \neq 0$. The residual vanishes everywhere if and only if $\lambda = 4k^2$; no division by a possibly zero cosine is needed.

Step 3: Initial profile. At $t = 0$, the condition is $A \cos(kx) = 3 \cos(2x)$ for every real x . Evaluation at $x = 0$ gives $A = 3$. Taking two x -derivatives of this identity at 0 gives $-Ak^2 = -12$, so $k^2 = 4$. Since $k > 0$, $k = 2$, and $\lambda = 16$. Therefore

$$u(x, t) = 3e^{-16t} \cos(2x).$$

For this function $u_t = -48e^{-16t} \cos(2x) = 4u_{xx}$, and the initial profile is recovered at $t = 0$. The function is smooth for all real x, t ; in particular it works for forward time $t \geq 0$.

Step 4: Amount of data. The profile prescribes a value for every x along $t = 0$. The single condition $u(0, 0) = 3$ fixes only A , leaving any $k > 0$ with $\lambda = 4k^2$ possible. The full profile selects one candidate in the stated family; no uniqueness assertion about all possible PDE solutions has been proved.

Question 10

Let $a \in \mathbb{R}$ be a fixed parameter in the equation

$$(a - 1)y'' + (a^2 - 1)(y')^2 + (a + 1)y' = x.$$

The proposed initial data are

$$y(0) = 0, \quad y'(0) = 0.$$

Tasks

1. Classify the equation's order and linearity for every real value of a . Simplify the exceptional cases before classifying them.
2. For each parameter value that makes the equation linear, find all solutions satisfying the proposed data using elementary integration.
3. For $a = 1$, decide whether $y'(0) = 0$ supplies independent information beyond the equation. What happens if this condition is replaced by $y'(0) = 1$?
4. Explain why the mere appearance of the symbol y'' in a parameterized formula does not guarantee that every member is a second-order equation.

Question 10 – Solution

Strategy. Determine when the leading coefficient or the nonlinear coefficient vanishes. Classify only after substituting those exceptional parameter values.

Step 1: Exhaustive classification. Factor $a^2 - 1 = (a - 1)(a + 1)$.

a	Simplified equation	Classification
1	$2y' = x$	First order, linear
-1	$-2y'' = x$	Second order, linear
$a \neq \pm 1$	Original equation	Second order, nonlinear

For $a \neq \pm 1$, both the y'' coefficient and the $(y')^2$ coefficient are nonzero. Both linear exceptional equations are nonhomogeneous because their right-hand sides are not identically zero.

Step 2: Solve when $a = 1$. From $y' = x/2$,

$$y = \frac{x^2}{4} + C, \quad y(0) = 0 \implies C = 0.$$

Thus $\boxed{y = x^2/4 \text{ on } \mathbb{R}}$. Its derivative at 0 is 0, as required, and $2y' = x$ verifies the equation directly.

Step 3: Solve when $a = -1$. Two integrations of $y'' = -x/2$ give

$$y' = -\frac{x^2}{4} + C_1, \quad y = -\frac{x^3}{12} + C_1x + C_2.$$

The data yield $C_1 = C_2 = 0$, so $\boxed{y = -x^3/12 \text{ on } \mathbb{R}}$. Indeed $-2y'' = x$ and both initial values are 0.

Step 4: Redundant or incompatible data. For $a = 1$, the equation at $x = 0$ already forces $2y'(0) = 0$. Thus the stated slope condition is redundant; replacing it with $y'(0) = 1$ makes the problem inconsistent. The coefficient $a - 1$ vanishes at $a = 1$, removing y'' entirely. Order is determined by derivatives with nonzero coefficients in the actual equation, not by unsimplified notation.