

Question 9

Let $u_t = u_{xx}$ on $0 < x < \pi$, with zero endpoint values and $u(x, 0) = \sin x + \sin(2x)$. A proposed solution strategy assumes that the complete answer is one product $X(x)T(t)$ because the PDE is separable.

Tasks

1. Derive the exact solution and explain why separability of the PDE does not force every solution to be a single product.
2. Prove that a nonzero product solution must use a spatial eigenfunction. Explain how the argument handles zeros of X or T .
3. Test the exact solution with the two-by-two sample determinant at $x_1 = \pi/3$, $x_2 = 2\pi/3$, $t_1 = 0$, $t_2 = \log 2$. Compute it and use it to rule out one-product factorization.
4. Find the exact uniform error when $e^t u(x, t)$ is approximated by $\sin x$, and a sufficient time for that error to be at most $\varepsilon \in (0, 1)$. Sketch normalized profiles for second-mode ratios 1, 1/2, 1/10 and zero.

Question 9 – Solution

Strategy. An eigenfunction expansion can become nearly one mode without ever being exactly one product at a finite time.

Step 1: Superpose the two decays. The eigenvalues are one and four, so

$$u = e^{-t} \sin x + e^{-4t} \sin(2x).$$

The PDE and every datum follow term by term. Separation constructs building blocks; linearity allows their sums, whose distinct temporal factors need not combine into one product.

Step 2: Derive the restriction on a product. If $u = XT$ is a nonzero classical product, then $XT' = X''T$. Choose a time with $T \neq 0$ and points where $X \neq 0$. The ratios satisfy $T'/T = X''/X = -\lambda$, independent of both variables. The differential identity $X'' = -\lambda X$ extends across spatial zeros by continuity; the temporal ODE $T' = -\lambda T$ extends likewise. Zero boundary values give the Dirichlet eigenproblem. The trivial zero product is separate. Division at zeros is unnecessary and cannot create extra eigenfunctions.

Step 3: Use a factorization invariant. Every product has determinant

$$u(x_1, t_1)u(x_2, t_2) - u(x_1, t_2)u(x_2, t_1) = 0.$$

For the stated points, the four entries are

$$u(x_1, 0) = \sqrt{3}, \quad u(x_2, 0) = 0, \quad u(x_1, \log 2) = \frac{9\sqrt{3}}{32}, \quad u(x_2, \log 2) = \frac{7\sqrt{3}}{32}.$$

Their determinant is $\boxed{21/32 \neq 0}$, proving that the exact field is not one product.

Step 4: Quantify approximate separation. The normalized field is $e^t u = \sin x + e^{-3t} \sin(2x)$, so

$$\|e^t u - \sin x\|_\infty = e^{-3t}, \quad t \geq \frac{1}{3} \log(1/\varepsilon) \implies \|e^t u - \sin x\|_\infty \leq \varepsilon.$$

Equality in the norm follows at a point where $|\sin(2x)| = 1$. The plotted ratios correspond to $t = 0, (\log 2)/3, (\log 10)/3$ and the limiting profile. Approximation by a surviving mode does not retroactively justify a single-product initial ansatz.

