

Question 7

On $0 < x < \pi$, compare the heat equation $H_t = H_{xx}$ with the wave equation $W_{tt} = W_{xx}$. Both have zero endpoint displacement for positive time and the initial interior profile $f(x) = 1$; the wave has zero initial velocity. Use L^2 convergence for the initial displacement. For the wave, use the reflected bounded function obtained from the odd, 2π -periodic extension F of f , not a presumed finite-energy solution.

Tasks

1. Derive the common sine coefficients and the heat and wave series, stating the appropriate interpretation of each.
2. Use $W(x, t) = \frac{1}{2}[F(x - t) + F(x + t)]$ to determine the wave profile for $0 < t < \pi/2$, including values at its two fronts.
3. Explain the contrast with heat smoothing, and why neither field can converge uniformly to 1 over the whole open interval as $t \downarrow 0$. Sketch both profiles at $t = \pi/4$.
4. Compute the energy of the first N wave modes and show why this problem has no finite-energy wave solution with those Dirichlet initial data. Explain what remains valid in the reflected interpretation.

Question 7 – Solution

Strategy. The same initial Fourier coefficients can lead to a smooth heat field and a discontinuous wave field; convergence alone does not guarantee finite energy.

Step 1: Compute and evolve the coefficients. The coefficients are $b_n = 2(1 - (-1)^n)/(n\pi)$, so

$$H = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{e^{-n^2 t} \sin(nx)}{n}, \quad W = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nt) \sin(nx)}{n}.$$

Both recover f in L^2 as $t \downarrow 0$. Heat derivatives converge uniformly for every $t \geq \tau > 0$. The wave series represents the reflected function in L^2 and takes average values at jumps; its zero initial time derivative is understood in distributions.

Step 2: Track the two reflected fronts. The extension is $F(s) = \text{sgn}(\sin s)$ away from multiples of π , with value zero at those multiples. For $0 < t < \pi/2$,

$$W(x, t) = \begin{cases} 0, & 0 < x < t \text{ or } \pi - t < x < \pi, \\ 1, & t < x < \pi - t, \\ 1/2, & x = t \text{ or } x = \pi - t. \end{cases}$$

Each endpoint trace is zero. Reflection has produced two moving jumps, not a gradually smoothing profile.

Step 3: Compare smoothing and initial convergence. For $t > 0$, the heat field is smooth and $0 < H < 1$ in the interior, by diffusion and the maximum principle. The wave instead has the step profile above. For any positive time approaching zero, the wave's zero strips give sup-norm error one; the heat's continuous zero endpoint trace makes its supremum error also one over the open interval. Thus L^2 initial convergence does not imply uniform convergence across the incompatible endpoint layer.

Step 4: Test energy rather than assume it. The partial wave energy is

$$E_N = \frac{\pi}{4} \sum_{n \leq N} n^2 b_n^2 = \frac{4}{\pi} \#\{n \leq N : n \text{ odd}\} \rightarrow \infty.$$

No finite-energy Dirichlet wave has this initial profile; it is not in the zero-trace energy space. The reflected bounded field still solves the wave equation distributionally, with the stated initial displacement and zero initial velocity in the corresponding weak sense. At a moving jump, its derivatives need not be ordinary square-integrable functions.

