

Question 6

Consider diffusion with variable spatial coefficient:

$$u_t = (x^2 u_x)_x, \quad 1 < x < e, \quad u(1, t) = u(e, t) = 0,$$

$$u(x, 0) = x^{-1/2} [\sin(\pi \log x) + 2 \sin(2\pi \log x)].$$

Use $s = \log x$ and $U(s, t) = u(e^s, t)$.

Tasks

1. Derive the transformed PDE and find a substitution $U = e^{-s/2}V$ that removes its first spatial derivative.
2. Derive the original spatial eigenfunctions, eigenvalues and their $L^2(1, e; dx)$ norms. Construct the complete solution.
3. Derive the coefficient formula for general initial data in these eigenfunctions. Explain why ordinary unweighted sine projection in s of $u(e^s, 0)$ is incorrect.
4. Verify the original PDE and data and derive the energy identity $\frac{1}{2}(d/dt) \int_1^e u^2 dx = - \int_1^e x^2 u_x^2 dx$. Check it on each separated eigenfunction.

Question 6 – Solution

Strategy. Changing coordinates changes the differential operator and the integration measure; a further amplitude factor restores ordinary sine orthogonality.

Step 1: Transform the operator, not just the variable. The chain rule gives $u_x = U_s/x$ and $(x^2 u_x)_x = U_{ss} + U_s$. With $U = e^{-s/2}V$, direct differentiation yields

$$V_t = V_{ss} - \frac{1}{4}V, \quad 0 < s < 1, \quad V(0, t) = V(1, t) = 0.$$

Ignoring U_s would shift every eigenvalue incorrectly.

Step 2: Recover the spectrum and field. The sine modes give

$$X_n(x) = x^{-1/2} \sin(n\pi \log x), \quad \lambda_n = (n\pi)^2 + \frac{1}{4}.$$

Since $dx = e^s ds$, the two $x^{-1/2}$ factors cancel the measure in a product:

$$\int_1^e X_m X_n dx = \int_0^1 \sin(m\pi s) \sin(n\pi s) ds.$$

Thus the squared norms are $1/2$, with zero cross products. The solution is

$$u = e^{-\lambda_1 t} X_1 + 2e^{-\lambda_2 t} X_2.$$

Step 3: Use the correct projection. For initial data f ,

$$b_n = 2 \int_1^e f(x) x^{-1/2} \sin(n\pi \log x) dx = 2 \int_0^1 e^{s/2} f(e^s) \sin(n\pi s) ds.$$

It is the transformed field $V(s, 0) = e^{s/2} f(e^s)$ that has ordinary sine coefficients. For example, the ungauged sines are not orthogonal in dx :

$$\int_1^e \sin(\pi \log x) \sin(2\pi \log x) dx = -\frac{4\pi^2(e+1)}{(1+\pi^2)(1+9\pi^2)} \neq 0.$$

Step 4: Verify diffusion and dissipation. Each X_n satisfies $-(x^2 X_n')' = \lambda_n X_n$ and vanishes at both endpoints. The temporal factors therefore verify the original PDE and exactly recover the supplied initial coefficients. Integration by parts gives $\frac{1}{2}(d/dt) \int u^2 = [u x^2 u_x]_1^e - \int x^2 u_x^2 = -\int x^2 u_x^2$. On a mode $a e^{-\lambda_n t} X_n$, both sides equal $-\lambda_n a^2 e^{-2\lambda_n t}/2$, since $\int x^2 X_n'^2 = \lambda_n \int X_n^2 = \lambda_n/2$.