

Question 5

On the square $0 < x, y < \pi$, let

$$U_t = U_{xx} + U_{yy}, \quad U = 0 \text{ on every edge}, \quad U(x, y, 0) = \sin x \sin(2y) + 2 \sin(2x) \sin y.$$

Tasks

1. Derive the product eigenfunctions and explain why the two active products have the same decay rate. Construct U .
2. Determine whether U is separable as $T(t)F(x, y)$ and whether its nonzero spatial profile can be written as $X(x)Y(y)$. Prove both conclusions.
3. Find the interior zero-temperature curve, its limiting edge endpoints and the signs on either side. Sketch that geometry.
4. Replace the initial coefficients by unknown A, B . Explain why one fixed-point time trace cannot distinguish them, but traces at $(\pi/2, \pi/4)$ and $(\pi/4, \pi/2)$ can. Give exact recovery formulas.

Question 5 – Solution

Strategy. A repeated eigenvalue allows a common time factor without forcing the spatial combination to be a single coordinate product.

Step 1: Identify the repeated eigenvalue. Zero edges give $\sin(mx)\sin(ny)$ with eigenvalues $m^2 + n^2$. The pairs (1, 2) and (2, 1) both give five. Therefore

$$U = e^{-5t} [\sin x \sin(2y) + 2 \sin(2x) \sin y].$$

Every edge and the initial field are correct, and applying the Laplacian multiplies the bracket by -5 .

Step 2: Distinguish two meanings of separation. The displayed formula is exactly one product $T(t)F(x, y)$. It is not one product $T(t)X(x)Y(y)$. Indeed, the spatial slice at $y = \pi/2$ is $2 \sin(2x)$, whereas at $y = \pi/4$ it is $\sin x + \sqrt{2} \sin(2x)$. These two nonzero slices are not proportional by sine orthogonality, as they would have to be for $X(x)Y(y)$.

Step 3: Determine the nodal geometry. Factor the bracket as $2 \sin x \sin y (\cos y + 2 \cos x)$. The first two factors are positive inside the square, so the interior zero set is

$$\cos y = -2 \cos x, \quad \pi/3 < x < 2\pi/3.$$

Its limiting endpoints are $(\pi/3, \pi)$ and $(2\pi/3, 0)$. At fixed y , the field is positive for $x < \arccos(-\cos y/2)$ and negative to its right. The zero curve is unchanged by the positive decay factor.

Step 4: Resolve the equal-rate inverse problem. At any one sensor, the entire trace has the form $e^{-5t}(A\alpha + B\beta)$ with fixed spatial factors α, β ; it supplies only one scalar equation. At the two specified sensors the unwanted product vanishes and the wanted product equals one. Hence

$$A = e^{5t}U(\pi/2, \pi/4, t), \quad B = e^{5t}U(\pi/4, \pi/2, t).$$

Distinct time samples at one point cannot separate equal decay rates; spatially independent observations can.

