

Question 4

On $0 < x < 1$, compare $H_t = H_{xx}$ and $W_{tt} = W_{xx}$ with the moving boundary values

$$H(0, t) = W(0, t) = t, \quad H(1, t) = W(1, t) = 2t.$$

Take $H(x, 0) = W(x, 0) = 0$ and $W_t(x, 0) = 1 + x$.

Tasks

1. For general boundary values $A(t), B(t)$, derive the transformed PDE and initial data after subtracting $\ell(x, t) = (1 - x)A(t) + xB(t)$ in each problem.
2. For the specified data, decide whether the affine lifting itself solves each original PDE.
3. Construct the complete heat solution using a stationary correction to its transformed forcing and a sine expansion. Derive the required coefficients.
4. Verify the reconstructed original data and find the long-time heat offset relative to the moving line. Contrast it with the wave field and state the heat release-corner regularity caveat.

Question 4 – Solution

Strategy. Homogenizing boundary values creates a source involving the first or second time derivative of the lifting, depending on the PDE.

Step 1: Transform the PDE and every initial datum. Since $\ell_{xx} = 0$, the zero-endpoint fields $v = H - \ell$ and $z = W - \ell$ obey

$$v_t - v_{xx} = -\ell_t, \quad z_{tt} - z_{xx} = -\ell_{tt}.$$

Their initial displacements are the respective original profiles minus $\ell(x, 0)$; the wave initial velocity is the original velocity minus $\ell_t(x, 0)$. Transforming only the endpoints would omit essential source and initial terms.

Step 2: Test the actual moving line. Here $\ell = t(1 + x)$, so $\ell_t = 1 + x$ but $\ell_{tt} = \ell_{xx} = 0$. It fails the heat PDE but solves the wave PDE. The wave's transformed displacement and velocity are both zero, giving

$$W(x, t) = t(1 + x).$$

The heat correction has $v_t - v_{xx} = -(1 + x)$ and $v(x, 0) = 0$.

Step 3: Solve the forced heat correction. Seek $P'' = 1 + x$ with $P(0) = P(1) = 0$. Integration gives

$$P(x) = \frac{x^2}{2} + \frac{x^3}{6} - \frac{2x}{3} = \frac{x(x-1)(x+4)}{6}.$$

The sine coefficients of $1 + x$ are $a_n = 2[1 - 2(-1)^n]/(n\pi)$. Twice integrating P'' against a sine gives $p_n = -a_n/(n\pi)^2$. Thus

$$H = t(1 + x) + P(x) + \sum_{n \geq 1} \frac{2[1 - 2(-1)^n]}{(n\pi)^3} e^{-(n\pi)^2 t} \sin(n\pi x).$$

Step 4: Verify and interpret the offset. At both ends, P and every sine vanish. At $t = 0$ the summable sine coefficients reconstruct $-P$ uniformly, giving zero initial temperature. For $t > 0$, Gaussian damping permits differentiated convergence; substitution verifies $H_t = H_{xx}$. The initial flat field has $H_{xx} = 0$ while the boundary time derivatives are one and two, so joint classical corner smoothness is not asserted. Finally $H - \ell \rightarrow P < 0$ on $0 < x < 1$, uniformly on the closed interval, whereas $W - \ell = 0$ at all times. An identical boundary lifting can have different physical consequences in the two PDEs.