

Question 3

Consider the separated spatial problem

$$-X'' = \lambda X, \quad 0 < x < L, \quad X(0) = 0, \quad X'(L) + hX(L) = 0,$$

where $L > 0$ and $h \in \mathbb{R}$. The same boundary conditions are imposed on the heat equation $u_t = \kappa u_{xx}$ and the wave equation $v_{tt} = c^2 v_{xx}$, with $\kappa, c > 0$.

Tasks

1. Derive the energy identity for λ . Use $X(L)^2 \leq L \int_0^L X'^2 dx$ to identify when every eigenvalue is positive and when a zero eigenvalue occurs.
2. For $h < -1/L$, derive the negative-eigenvalue equation and prove there is exactly one negative eigenvalue.
3. Choose $L = 1$ and $h = -\coth 1$. Construct a heat solution and a wave solution with initial profile $\sinh x$, taking zero initial wave velocity.
4. Explain why these solutions invalidate a rule that separated heat modes always decay and separated wave modes always oscillate. Identify the boundary term responsible.

Question 3 – Solution

Strategy. The admissible spatial spectrum depends on the boundary operator, including the sign of a Robin coefficient.

Step 1: Include the Robin boundary term. Multiplication by X and integration by parts give

$$\lambda \int_0^L X^2 dx = \int_0^L X'^2 dx + hX(L)^2.$$

For $h \geq 0$ the right side is positive for nonzero X with $X(0) = 0$. For $-1/L < h < 0$, the supplied trace inequality bounds it below by $(1 + hL) \int X'^2 > 0$. At $h = -1/L$ it is nonnegative, with equality precisely for an affine function through zero; $X = x$ is the zero mode. Conversely, solving $X'' = 0$ shows a nonzero zero mode occurs only at $h = -1/L$.

Step 2: Find the negative mode. For $\lambda = -\sigma^2$, $\sigma > 0$, the left condition gives $X = A \sinh(\sigma x)$. The right condition is

$$\sigma \coth(\sigma L) = -h.$$

The left side increases strictly from $1/L$ to infinity: with $z = \sigma L$, the derivative of $z \coth z$ is $(\sinh z \cosh z - z)/\sinh^2 z > 0$. Thus there is exactly one positive σ when $-h > 1/L$, and none otherwise. This counts all negative eigenvalues because every such eigenfunction must have the displayed hyperbolic form.

Step 3: Build the two growing examples. For the specified L, h , $\sigma = 1$ and $\lambda = -1$. Hence

$$u(x, t) = e^{\kappa t} \sinh x, \quad v(x, t) = \cosh(ct) \sinh x.$$

Their spatial second derivatives equal the fields; their time derivatives satisfy the respective PDEs. The left traces are zero, and at $x = 1$ the Robin expression is proportional to $\cosh 1 - \coth 1 \sinh 1 = 0$. The initial profiles are $\sinh x$ and the initial wave velocity is zero.

Step 4: Locate the failure of the rule. Heat factors satisfy $a' = -\kappa \lambda a$; wave factors satisfy $b'' + c^2 \lambda b = 0$. A negative eigenvalue therefore gives growth rather than decay or pure oscillation. In the spatial energy identity, $hX(L)^2$ is negative here. The usual nonnegative energy form for dissipative or fixed boundaries cannot be assumed for this Robin condition. The counterexample changes the spectrum, not the separation calculation.