

Question 10

A unit fixed-end string with unit tension and density is released from rest with $u(x, 0) = x(1 - x)$. Let u_N retain only sine modes with index $n \leq N$, where $N \geq 1$ is an integer, from the exact finite-energy solution. Define the error energy

$$E_{\text{err}}(t) = \frac{1}{2} \int_0^1 [(u_t - u_{N,t})^2 + (u_x - u_{N,x})^2] dx.$$

Tasks

1. Derive the Fourier solution and its total energy. State the regularity caveat at the initial boundary corners.
2. Derive an exact series for E_{err} and prove the all-time estimate $E_{\text{err}} \leq 16/(3\pi^4 N^3)$.
3. Prove the all-time displacement estimate $\|u - u_N\|_{\infty} \leq 4/(\pi^3 N^2)$. Find a sufficient integer N for error at most 10^{-3} everywhere.
4. Find a sufficient integer N for relative energy error at most 10^{-4} . Explain why neither estimate depends on elapsed time and why wave evolution supplies no heat-like smoothing factor.

Question 10 – Solution

Strategy. Orthogonality gives a conserved tail energy, while absolute Fourier tails give a different uniform displacement certificate.

Step 1: Construct the solution and normalize energy. Two integrations by parts give

$$b_n = 2 \int_0^1 x(1-x) \sin(n\pi x) dx = \frac{4(1 - (-1)^n)}{(n\pi)^3}.$$

Thus

$$u(x, t) = \frac{8}{\pi^3} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{\sin(n\pi x) \cos(n\pi t)}{n^3}, \quad E = \frac{1}{2} \int_0^1 (1-2x)^2 dx = \frac{1}{6}.$$

Displacement and first-derivative series converge uniformly, and the PDE has a finite-energy interpretation. A globally C^2 corner solution is impossible: fixed endpoints would have $u_{tt} = 0$ there, while the initial second spatial derivative is -2 .

Step 2: Bound the conserved energy tail. Each omitted mode contributes $(n\pi)^2 b_n^2/4$, independent of time. Orthogonality therefore gives

$$E_{\text{err}} = \frac{16}{\pi^4} \sum_{\substack{n > N \\ n \text{ odd}}} \frac{1}{n^4} \leq \frac{16}{\pi^4} \sum_{n > N} \frac{1}{n^4} \leq \boxed{\frac{16}{3\pi^4 N^3}}.$$

The last inequality is the decreasing-function integral bound $\sum_{n > N} n^{-4} \leq \int_N^\infty s^{-4} ds$.

Step 3: Certify displacement independently. Using $|\sin|, |\cos| \leq 1$ and the same tail comparison,

$$\|u - u_N\|_\infty \leq \frac{8}{\pi^3} \sum_{n > N} \frac{1}{n^3} \leq \frac{4}{\pi^3 N^2}.$$

This holds simultaneously for every x and every t . For tolerance $\delta = 10^{-3}$, it suffices to take $N \geq \sqrt{4/(\pi^3 \delta)} \approx 11.36$, so $\boxed{N = 12}$ suffices. The bound includes even indices for convenience; it is sufficient, not a claim of the smallest possible retained set.

Step 4: Choose an energy-based cutoff. Since $E = 1/6$, $E_{\text{err}}/E \leq 32/(\pi^4 N^3)$. For tolerance $\eta = 10^{-4}$, it suffices that

$$N \geq \left(\frac{32}{\pi^4 \eta} \right)^{1/3} \approx 14.87, \quad \boxed{N = 15}.$$

Each wave mode preserves its energy and merely oscillates. There is no factor decaying with $n^2 t$ as in diffusion, so waiting does not reduce the truncation's energy error. These two cutoffs measure different errors and need not agree.