

Question 8

A uniform string of length L , tension $T > 0$ and density $\mu > 0$ is fixed at both ends and carries a point mass $M > 0$ at its midpoint. Away from that mass, $u_{tt} = c^2 u_{xx}$ with $c = \sqrt{T/\mu}$. Displacement is continuous at $x = L/2$, and the mass obeys

$$Mu_{tt}(L/2, t) = T[u_x(L/2+, t) - u_x(L/2-, t)].$$

Tasks

1. For a normal mode $u = \phi(x) \cos(\omega t)$, derive the interface condition and classify antisymmetric modes about the midpoint.
2. Derive the frequency equation for symmetric modes using $z = \omega L/(2c)$ and $\alpha = M/(\mu L)$. Prove there is exactly one positive symmetric root in each interval $(j\pi, j\pi + \pi/2)$, $j = 0, 1, \dots$
3. Find the limiting lowest symmetric frequency as $M \downarrow 0$ and its leading behavior as $M \rightarrow \infty$. Interpret the latter as an effective spring-mass system.
4. Write the total conserved energy, including the attached mass, and prove conservation by checking the interface terms.

Question 8 – Solution

Strategy. The midpoint mass changes the dynamic matching condition, but modes with a node there do not move the added mass.

Step 1: Separate and use symmetry. On each half, $\phi'' + k^2\phi = 0$, $k = \omega/c$, and the mass equation becomes

$$T(\phi'_+ - \phi'_-) + M\omega^2\phi(L/2) = 0.$$

Antisymmetry gives $\phi(L/2) = 0$ and matching slopes. The fixed endpoints then yield

$$\phi(x) = A \sin(2m\pi x/L), \quad \omega = 2m\pi c/L, \quad m = 1, 2, \dots$$

These frequencies are independent of M .

Step 2: Derive and locate the symmetric frequencies. A symmetric shape has $\phi = A \sin(kx)$ on the left and $\phi = A \sin(k(L-x))$ on the right. Its slope jump is $-2Ak \cos z$ and midpoint value is $A \sin z$. Substitution gives

$$2Tk \cos z = Mc^2 k^2 \sin z, \quad \cot z = \alpha z, \quad \alpha = \frac{M}{\mu L}.$$

On $(j\pi, j\pi + \pi/2)$, $\cot z - \alpha z$ decreases strictly from positive infinity to a negative value. It has exactly one root. On the intervening intervals the cotangent is negative, so no positive root occurs there. Values $\sin z = 0$ do not solve the original matching equation. There is no nonzero static mode with both endpoints fixed.

Step 3: Interpret the lowest root. For the root $z_0 \in (0, \pi/2)$, $\alpha \downarrow 0$ gives $z_0 \rightarrow \pi/2$, recovering $\omega_0 \rightarrow \pi c/L$. As $\alpha \rightarrow \infty$, $z_0 \rightarrow 0$ and $z \cot z \rightarrow 1$ gives $\alpha z_0^2 \rightarrow 1$. Thus

$$\omega_0 \sim 2\sqrt{\frac{T}{ML}} \quad (M \rightarrow \infty).$$

A small midpoint deflection d gives each half-string slope magnitude $2d/L$; their restoring forces add to $4Td/L$. The effective stiffness is $4T/L$, consistent with the displayed mass-spring frequency.

Step 4: Include the concentrated kinetic energy. For piecewise smooth motion, the energy is

$$E = \frac{1}{2} \int_0^L (\mu u_t^2 + T u_x^2) dx + \frac{M}{2} u_t(L/2, t)^2.$$

Integration by parts on each half leaves $T u_t(L/2)(u_x^- - u_x^+)$; the fixed-end contributions vanish. The point-mass term differentiates to $M u_t(L/2) u_{tt}(L/2) = T u_t(L/2)(u_x^+ - u_x^-)$. They cancel, so $\boxed{E' = 0}$. Omitting the mass term would falsely suggest loss or gain of energy at the interface.