

Question 7

A unit fixed-end string with speed one is known to involve only modes $1, \dots, N$:

$$u(x, t) = \sum_{n=1}^N \left[f_n \cos(n\pi t) + \frac{g_n}{n\pi} \sin(n\pi t) \right] \sin(n\pi x).$$

A displacement sensor at $x = s \in (0, 1)$ records $h_s(t) = u(s, t)$ for $0 \leq t \leq 2$.

Tasks

1. Derive recovery formulas for f_n, g_n from the trace whenever the n th mode is visible.
2. For a rational sensor position $s = p/q$ in lowest terms, identify exactly the invisible modes. Explain why continuous-time sampling cannot recover those modes.
3. Use sensors at $s = 1/2$ and $s = 1/3$. Determine their common invisible modes and the largest N for which the entire stated finite-mode state is uniquely recoverable.
4. Prove that an irrational sensor sees every individual mode but can be arbitrarily poorly conditioned as N grows. You may use the pigeonhole fact that for every integer $Q \geq 1$ there are integers $1 \leq n \leq Q, k$ with $|ns - k| \leq 1/Q$.

Question 7 – Solution

Strategy. Time orthogonality separates frequencies, but it cannot restore a spatial mode whose node lies at the sensor.

Step 1: Project the temporal record. On $[0, 2]$, distinct integer-frequency sines and cosines are orthogonal and each squared norm is one. Consequently

$$f_n = \frac{\int_0^2 h_s(t) \cos(n\pi t) dt}{\sin(n\pi s)}, \quad g_n = \frac{n\pi \int_0^2 h_s(t) \sin(n\pi t) dt}{\sin(n\pi s)}$$

whenever $\sin(n\pi s) \neq 0$. These recover both initial displacement and velocity coefficients.

Step 2: Identify rational-position blindness. For $s = p/q$ in lowest terms, $\sin(n\pi s) = 0$ exactly when q divides n . Every such mode contributes zero to $h_s(t)$ for every time and for every choice of its two coefficients. Thus even a perfect continuous trace cannot distinguish states that differ only in those modes.

Step 3: Combine the two sensors. The midpoint misses the even modes; the one-third sensor misses multiples of three. Their common invisible modes are exactly the multiples of six. For $N \leq 5$, each mode has a nonzero factor at at least one sensor, so the formulas recover the entire state. For $N \geq 6$, arbitrary sixth-mode displacement or velocity can be added without changing either record. The largest fully identifiable cutoff is therefore

$$N = 5.$$

Step 4: Distinguish visibility from stability. If s is irrational, no ns is an integer, so each denominator is nonzero. But the supplied approximation gives $|\sin(n\pi s)| = |\sin(\pi(ns - k))| \leq \pi/Q$. As $Q \rightarrow \infty$, the selected indices cannot stay in a finite set: each such index has a fixed positive distance from the integers. Hence arbitrarily large modes have arbitrarily small sensor factors. An error e_n in the cosine projection becomes $e_n/\sin(n\pi s)$ in f_n ; the sine projection has the additional factor $n\pi$ for g_n . Exact visibility at all modes does not provide a uniform noise bound.