

Question 6

A unit string with unit tension and density has fixed ends and is initially at rest. A spatially sinusoidal force is held constant for a chosen duration $\tau > 0$:

$$u_{tt} - u_{xx} = F_0 \sin(\pi x) \chi_{(0,\tau)}(t), \quad u(x, 0) = u_t(x, 0) = 0, \quad F_0 > 0.$$

The indicator is one during the pulse and zero otherwise; isolated switch-time values do not affect the motion.

Tasks

1. Derive the modal displacement during and after the pulse, imposing continuity of displacement and velocity at switch-off.
2. Determine every positive duration that leaves the string exactly at rest after switch-off. Prove that both state variables vanish for those durations.
3. Compute the residual energy as a function of τ and verify it equals the net work of the applied force.
4. Find the durations that maximize residual energy. Plot its normalized dependence on pulse duration and explain how a force of one sign can do zero net work over a nontrivial pulse.

Question 6 – Solution

Strategy. Switch-off is a delayed negative step in forcing; controlling pulse duration can cancel the entire excited state.

Step 1: Superpose the two forcing steps. Only the first spatial mode is driven. Writing $u = q(t) \sin(\pi x)$ gives $q'' + \pi^2 q = F_0 \chi_{(0,\tau)}$ with zero initial data. Hence

$$q(t) = \frac{F_0}{\pi^2} \begin{cases} 1 - \cos(\pi t), & 0 \leq t \leq \tau, \\ \cos(\pi(t - \tau)) - \cos(\pi t), & t \geq \tau. \end{cases}$$

The two expressions and their first derivatives agree at $t = \tau$; only acceleration can jump. They verify the forcing on each open time interval.

Step 2: Find exact cancellation durations. For $t \geq \tau$, the cosine difference becomes

$$q(t) = \frac{2F_0}{\pi^2} \sin \frac{\pi\tau}{2} \sin(\pi(t - \tau/2)).$$

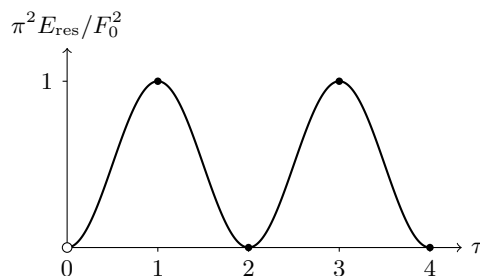
Its displacement and velocity vanish identically precisely when $\sin(\pi\tau/2) = 0$, giving $\tau = 2m, m = 1, 2, \dots$. Equivalently, the switch-off state $q(\tau) = F_0(1 - \cos \pi\tau)/\pi^2$, $q'(\tau) = F_0 \sin \pi\tau/\pi$ is zero exactly for those durations.

Step 3: Evaluate residual energy and work. After switch-off,

$$E_{\text{res}} = \frac{1}{4}(q'^2 + \pi^2 q^2) = \frac{F_0^2}{\pi^2} \sin^2 \frac{\pi\tau}{2}.$$

During forcing, the power is $\int_0^1 F_0 \sin(\pi x) u_t dx = F_0 q'/2$. Its integral from zero to τ is $F_0 q(\tau)/2 = F_0^2(1 - \cos \pi\tau)/(2\pi^2)$, the same energy.

Step 4: Interpret the pulse-duration curve. The maximum is F_0^2/π^2 , attained for $\tau = 2m + 1, m \geq 0$. For an even-integer duration, positive work during motion in the force direction is canceled by negative work while the string moves against the force. A positive force need not supply positive power at every instant.



Open dot: the excluded zero-duration limit.