

Question 5

A unit string with fixed ends, unit density and unit tension experiences uniform viscous drag:

$$u_{tt} + 2\beta u_t = u_{xx}, \quad \beta = 2\pi, \quad u(x, 0) = \sin(\pi x) + \sin(2\pi x) + \sin(3\pi x), \quad u_t(x, 0) = 0.$$

Tasks

1. Derive the modal equations and classify each of the three active modes as over-, critically or under-damped.
2. Construct the full solution, satisfying both initial conditions for every mode.
3. Derive the energy dissipation law for $E = \frac{1}{2} \int_0^1 (u_t^2 + u_x^2) dx$. Explain why energy is nonincreasing even when a modal displacement oscillates.
4. Find the leading long-time displacement profile and its exact exponential decay rate. More generally, state when all positive string modes are underdamped.

Question 5 – Solution

Strategy. The same drag rate competes with different natural frequencies, so one string can exhibit all three damping regimes at once.

Step 1: Classify the modal roots. With $u = \sum q_n(t) \sin(n\pi x)$, each coefficient satisfies $q_n'' + 2\beta q_n' + (n\pi)^2 q_n = 0$. For $\beta = 2\pi$, the discriminant sign is the sign of $4 - n^2$. Thus mode one is overdamped, mode two is critical, and mode three is underdamped.

Step 2: Impose both initial conditions. The required coefficient functions, all with $q_n(0) = 1$ and $q_n'(0) = 0$, are

$$\begin{aligned} q_1 &= e^{-2\pi t} \left[\cosh(\sqrt{3}\pi t) + \frac{2}{\sqrt{3}} \sinh(\sqrt{3}\pi t) \right], \\ q_2 &= e^{-2\pi t} (1 + 2\pi t), \\ q_3 &= e^{-2\pi t} \left[\cos(\sqrt{5}\pi t) + \frac{2}{\sqrt{5}} \sin(\sqrt{5}\pi t) \right]. \end{aligned}$$

Then $u = \sum_{n=1}^3 q_n(t) \sin(n\pi x)$ satisfies the damped PDE and both endpoints. The added sinh, linear and sine terms are necessary to make the initial velocities zero.

Step 3: Account for dissipated energy. Differentiate E and integrate $u_x u_{xt}$ by parts:

$$E' = [u_x u_t]_0^1 + \int_0^1 u_t (u_{tt} - u_{xx}) dx = \left[-2\beta \int_0^1 u_t^2 dx \leq 0 \right].$$

The boundary term is zero because fixed ends have zero velocity. Oscillation changes how energy is partitioned; drag still removes energy whenever the velocity is not identically zero.

Step 4: Identify the slow surviving mode. Expanding q_1 into two exponentials, its slower coefficient is $C = (1 + 2/\sqrt{3})/2$ and its slower decay rate is $\gamma = (2 - \sqrt{3})\pi$. The other two modes decay as $e^{-2\pi t}$ times bounded or linear factors. Therefore

$$e^{\gamma t} u(x, t) \longrightarrow C \sin(\pi x) \quad \text{uniformly for } 0 \leq x \leq 1.$$

For a nonnegative general drag parameter, mode n is underdamped when $\beta < n\pi$; all positive modes are underdamped exactly when $0 \leq \beta < \pi$. The threshold compares drag with the lowest frequency, not with the highest one.