

Question 4

A string of length L has tension $T > 0$, density $\mu > 0$ and speed $c = \sqrt{T/\mu}$. Its left end is fixed and its right end is free for $t > 0$:

$$u_{tt} = c^2 u_{xx}, \quad u(0, t) = 0, \quad u_x(L, t) = 0.$$

At release its displacement is $f(x) = hx/L$, $h > 0$, and its velocity is zero. Interpret this as a finite-energy problem, since the initial slope need not satisfy the newly released end condition.

Tasks

1. Derive the mixed-end eigenfunctions, frequencies and coefficients of f . Construct the solution.
2. Explain the initial corner incompatibility. Prove that the full state reverses sign after $2L/c$ and has smallest positive period $4L/c$.
3. Show that at $t = L/c$ the string is flat but is not at rest. Find its velocity in the open interval and verify the conserved energy.
4. For $L = c = h = 1$, sketch exact profiles at $t = 0, 1/2, 1$. Use the odd reflection at the fixed end and even reflection at the free end, and mark the moving corner.

Question 4 – Solution

Strategy. A free end selects half-integer modes; the release launches a corner rather than making the initial slope compatible retroactively.

Step 1: Construct the mixed-mode solution. The conditions $X(0) = 0$, $X'(L) = 0$ give $k_n = (2n+1)\pi/(2L)$, $n \geq 0$, and $\omega_n = ck_n$. Since $\int_0^L \sin^2(k_n x) dx = L/2$,

$$b_n = \frac{2}{L} \int_0^L \frac{hx}{L} \sin(k_n x) dx = \frac{8h(-1)^n}{(2n+1)^2 \pi^2}, \quad \boxed{u = \sum_{n \geq 0} b_n \sin(k_n x) \cos(\omega_n t)}.$$

Step 2: State the trace and recurrence precisely. Initially $f'(L) = h/L \neq 0$, so a solution with continuous u_x at the release corner cannot satisfy the new free-end condition there. The displayed finite-energy solution recovers f uniformly and zero initial velocity in L^2 ; its boundary condition has the usual weak/natural meaning at wavefront events. Every active frequency is an odd multiple of $\pi c/(2L)$. Thus both u and u_t change sign after $2L/c$ and return after $4L/c$. The nonzero lowest mode requires that full period, proving minimality.

Step 3: Resolve the flat but moving state. At $t = L/c$ every temporal cosine vanishes. For $0 < x < L$,

$$u_t(x, L/c) = -\frac{4hc}{L\pi} \sum_{n \geq 0} \frac{\sin((2n+1)\pi x/(2L))}{2n+1} = \boxed{-hc/L}.$$

Here $\sum_{n \geq 0} \sin((2n+1)z)/(2n+1) = \pi/4$ for $0 < z < \pi$, the sine series of a constant on that interval. The fixed-end velocity remains zero; the interior velocity trace is an L^2 statement, not corner continuity. Initially $E = Th^2/(2L)$. At this flat snapshot, the kinetic energy is $\mu L(hc/L)^2/2 = Th^2/(2L)$, so the energy is unchanged.

Step 4: Follow the released-end corner. Extend f oddly across zero and evenly across L , with period $4L$, and take the average of its two translates. For the stated units and $0 \leq t \leq 1$ this gives

$$\boxed{u(x, t) = \min(x, 1-t), \quad 0 \leq x \leq 1.}$$

For $0 < t < 1$, the corner is at $x = 1-t$: the left part retains slope one and the right part is flat and moving downward. The snapshots below use exact line segments.

