

Question 2

A uniform string of length L , tension $T > 0$ and density $\mu > 0$ has fixed endpoints and wave speed $c = \sqrt{T/\mu}$. It is plucked to height $h > 0$ at $x = a$, where $0 < a < L$, then released from rest:

$$f(x) = \begin{cases} hx/a, & 0 \leq x \leq a, \\ h(L-x)/(L-a), & a \leq x \leq L, \end{cases} \quad u(x, 0) = f(x), \quad u_t(x, 0) = 0.$$

Use the finite-energy solution of $u_{tt} = c^2 u_{xx}$.

Tasks

1. Derive the sine coefficients and the full solution. State in what sense the initial data and PDE are satisfied near the moving corners.
2. Set $a = L/3$. Identify precisely which modes are absent and compute the initial, hence conserved, energy.
3. Prove $u(x, L/c) = -f(L-x)$ and $u(x, 2L/c) = f(x)$. Include the velocity state at these times.
4. Let F be the odd, $2L$ -periodic extension of f . Use it to obtain exact piecewise-linear profiles at $t = 0, L/(2c), L/c$ for $L = c = h = 1$, $a = 1/3$, and sketch them without truncating a Fourier series.

Question 2 – Solution

Strategy. Fourier coefficients reveal the missing harmonics, while reflected traveling profiles preserve the corners exactly.

Step 1: Compute the pluck coefficients. Integrating separately on $[0, a]$ and $[a, L]$ gives

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx = \frac{2hL^2}{a(L-a)n^2\pi^2} \sin \frac{n\pi a}{L}.$$

Thus

$$u(x, t) = \sum_{n \geq 1} b_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}.$$

The $O(n^{-2})$ coefficients give uniform convergence of displacement. The initial velocity is zero in L^2 , and the solution has finite conserved energy. It solves the PDE weakly and classically away from the propagating slope corners; no globally C^2 solution is claimed.

Step 2: Identify the spectrum and energy. For $a = L/3$, $b_n = 9h \sin(n\pi/3)/(n^2\pi^2)$, so exactly the positive multiples of three are absent. Initially all energy is elastic:

$$E = \frac{T}{2} \int_0^L f'^2 dx = \frac{Th^2L}{2a(L-a)} = \boxed{\frac{9Th^2}{4L}} \quad (a = L/3).$$

The finite-energy wave evolution conserves this value.

Step 3: Verify the reflected and full returns. At $t = L/c$, each cosine is $(-1)^n$, and $\sin(n\pi(L-x)/L) = (-1)^{n+1} \sin(n\pi x/L)$. Hence $u(x, L/c) = -f(L-x)$. At $2L/c$ every cosine equals one, so $u = f$. The modal velocities vanish at both times, giving zero velocity in L^2 .

Step 4: Construct exact corner-preserving snapshots. The reflected d'Alembert form is $u(x, t) = \frac{1}{2}[F(x-ct) + F(x+ct)]$. Oddness and periodicity enforce both endpoints and recover both initial data. For the stated numbers, the half-time profile is

$$u(x, \tfrac{1}{2}) = \begin{cases} -3x/2, & 0 \leq x \leq 1/6, \\ 3x/4 - 3/8, & 1/6 \leq x \leq 5/6, \\ 3(1-x)/2, & 5/6 \leq x \leq 1. \end{cases}$$

The other two profiles are $f(x)$ and $-f(1-x)$; their corners are plotted at their exact positions.

