

Question 9

Let u be regular and harmonic on the unit disk. Its boundary temperature is known to have the form

$$g(\theta) = A + B \cos(N\theta), \quad N \geq 1,$$

where N is known and A, B are unknown. The temperature is measured on the circle $r = \rho$, where $0 < \rho < 1$.

Tasks

1. Derive exact recovery formulas for A, B from the measured angular mean and cosine coefficient. Specify the consistency conditions on the other measured Fourier coefficients.
2. For $\rho = 1/2$, $N = 6$, and measured trace $2 + 0.03 \cos(6\theta)$, recover the boundary data and the full field.
3. If the measured N th cosine coefficient has error at most η , find the sharp error bound for the recovered B . Derive the largest allowable mode number when the desired per-coefficient error is at most $\varepsilon > \eta$. Evaluate it for $\rho = 1/2, \eta = 10^{-4}, \varepsilon = 10^{-2}$.
4. Contrast forward boundary stability with inverse instability when arbitrarily high modes are allowed. Give an explicit sequence showing why small interior-circle errors need not imply small boundary errors.

Question 9 – Solution

Strategy. Harmonic continuation damps a mode by ρ^N ; inversion must divide by exactly that small factor.

Step 1: Recover the two parameters. The regular harmonic extension is $u = A + Br^N \cos(N\theta)$. For the measured trace $h(\theta) = u(\rho, \theta)$, define

$$m = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) d\theta, \quad c_N = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \cos(N\theta) d\theta.$$

Orthogonality gives

$$A = m, \quad B = \rho^{-N} c_N.$$

Consistency with the stated model requires every sine coefficient and every nonconstant cosine coefficient except c_N to vanish. Since $\rho > 0$, the two parameters are unique.

Step 2: Reconstruct the given field. Here $\rho^N = 1/64$, $m = 2$, and $c_6 = 0.03 = 3/100$, so $B = 48/25 = 1.92$. Thus

$$g(\theta) = 2 + \frac{48}{25} \cos(6\theta), \quad u(r, \theta) = 2 + \frac{48}{25} r^6 \cos(6\theta).$$

Substitution at $r = 1/2$ reproduces the measured coefficient $3/100$. The polar mode is a Cartesian harmonic polynomial and is regular at the center.

Step 3: Quantify inverse amplification. A coefficient error e , $|e| \leq \eta$, produces error $\rho^{-N}e$ in B . Consequently $|\delta B| \leq \eta \rho^{-N}$, with equality when $|e| = \eta$. The desired bound is met exactly when

$$N \leq \frac{\log(\varepsilon/\eta)}{\log(1/\rho)}, \quad N_{\max} = \left\lfloor \frac{\log(\varepsilon/\eta)}{\log(1/\rho)} \right\rfloor.$$

If this integer is zero, no positive mode qualifies. The given numbers give $N_{\max} = 6$: mode six has error bound 0.0064, whereas mode seven has 0.0128. This is a per-coefficient criterion, not a bound on a sum of many mode errors.

Step 4: Separate forward and inverse stability. For continuous boundary perturbations, the maximum principle gives $\|u_1 - u_2\|_{\infty, \text{disk}} \leq \|g_1 - g_2\|_{\infty, \partial \text{disk}}$. In contrast, take $g_n(\theta) = \cos(n\theta)$ and $u_n = r^n \cos(n\theta)$. Then $\|g_n\|_{\infty} = 1$ but $\|u_n(\rho, \cdot)\|_{\infty} = \rho^n \rightarrow 0$. Thus no fixed bound can control boundary sup-norm errors by measured-circle sup-norm errors across all modes, although each exact finite-mode inversion is unique.