

Question 10

Let $u_t = \kappa u_{xx}$ on $0 < x < L$, with fixed endpoint temperatures A, B , $\kappa, L > 0$, and stationary field $S = A + (B - A)x/L$. For smooth compatible data define the error energy $E_v = \frac{1}{2} \int_0^L (u - S)^2 dx$ and the raw quadratic quantity $E_u = \frac{1}{2} \int_0^L u^2 dx$. You may use the zero-endpoint Poincare inequality $\int v_x^2 \geq (\pi/L)^2 \int v^2$ and the parabolic maximum principle.

Tasks

1. Derive the energy identities for E_v and E_u , retaining all boundary terms. Prove the sharp bound $E_v(t) \leq e^{-2\lambda t} E_v(0)$, where $\lambda = \kappa \pi^2 / L^2$.
2. For $A = B = 1$ and $u(x, 0) = 1 - \varepsilon \sin(\pi x / L)$, $0 < \varepsilon \leq 1$, construct the solution. Prove that its raw quadratic quantity increases while its error energy decreases.
3. If $|u(x, 0) - S(x)| \leq C \sin(\pi x / L)$, $C > 0$, use comparison to prove a pointwise error bound for all later times. Give a sufficient time for uniform temperature error at most δ , where $0 < \delta < C$.
4. Give a sufficient time for reducing error energy to a fraction $\eta \in (0, 1)$ of its initial value, and explain why neither total heat nor raw quadratic energy alone measures distance from a maintained nonzero equilibrium.

Question 10 – Solution

Strategy. Dissipation applies naturally to deviations from the maintained equilibrium; the boundary reservoirs can increase the unshifted quantities.

Step 1: Identify which energy has no boundary work. For $v = u - S$, the endpoints vanish and $v_t = \kappa v_{xx}$. Therefore

$$E'_v = -\kappa \int_0^L v_x^2 dx \leq -2\lambda E_v, \quad E_v(t) \leq e^{-2\lambda t} E_v(0).$$

A first-sine error attains equality, so the rate is sharp. For the raw quantity,

$$E'_u = \kappa [uu_x]_0^L - \kappa \int_0^L u_x^2 dx.$$

The first term need not vanish with nonzero maintained boundary values.

Step 2: Exhibit simultaneous raw growth and error decay. The stated initial data give $u = 1 - \varepsilon e^{-\lambda t} \sin(\pi x/L)$. Then

$$E_u = \frac{L}{2} - \frac{2\varepsilon L}{\pi} e^{-\lambda t} + \frac{\varepsilon^2 L}{4} e^{-2\lambda t}, \quad E_v = \frac{\varepsilon^2 L}{4} e^{-2\lambda t}.$$

Writing $z = \varepsilon e^{-\lambda t} \in (0, 1]$ gives $E'_u = \lambda L z(2/\pi - z/2) > 0$, since $2/\pi > 1/2$. Meanwhile $E'_v = -2\lambda E_v < 0$. Heating by the reservoirs moves this field closer to equilibrium while increasing its raw quadratic quantity.

Step 3: Establish a pointwise stopping criterion. The barriers $\pm C e^{-\lambda t} \sin(\pi x/L)$ solve the homogeneous zero-endpoint heat equation and bound the initial error. Applying comparison to their differences with v on any finite time interval gives

$$|u - S| \leq C e^{-\lambda t} \sin(\pi x/L) \leq C e^{-\lambda t}.$$

Thus $t \geq \lambda^{-1} \log(C/\delta)$ suffices for uniform error at most δ . This conclusion uses the supplied pointwise initial bound, not merely an initial L^2 norm.

Step 4: Distinguish the two stopping criteria. The energy inequality gives $E_v(t)/E_v(0) \leq \eta$ once $t \geq (2\lambda)^{-1} \log(1/\eta)$ for nonzero initial error; zero error is already at equilibrium. The factor of two comes from squaring a decaying amplitude. Total heat can conceal spatial differences, and raw energy includes the maintained background and boundary work. Neither is a substitute for an explicit norm of $u - S$ when measuring equilibration.