

Question 9

A rod on $0 < x < 1$ obeys $u_t = u_{xx}$ with unknown constant boundary temperatures A, B . It is known that its full field has the form

$$u(x, t) = A + (B - A)x + Ce^{-\pi^2 t} \sin(\pi x).$$

Sensors at $x = 1/3, 2/3$ record $y_L(t), y_R(t)$. Set $m(t) = (y_L(t) + y_R(t))/2$ and let $\Delta > 0$ be known.

Tasks

1. Derive which combination of the readings identifies the stationary gradient immediately, even before the transient has decayed.
2. Use readings at times $0, \Delta$ to recover A, B, C uniquely. State the consistency condition on the differences of the two pairs of readings.
3. For $\Delta = (\log 2)/\pi^2$, use readings $(8/3, 10/3)$ at zero and $(13/6, 17/6)$ at Δ to find the complete field and verify the readings and boundary values.
4. Explain why one pair of simultaneous readings cannot identify all three parameters. For small Δ , quantify amplification of errors in the two averaged readings when estimating the mean of the boundary temperatures.

Question 9 – Solution

Strategy. Equal sine values at the two sensors let their difference cancel the transient; a second observation time separates the remaining steady and decaying parts.

Step 1: Cancel the equal transient values. At both sensor positions, $\sin(\pi x) = \sqrt{3}/2$. Thus

$$d := B - A = 3(y_R - y_L), \quad m(t) = S + Ke^{-\pi^2 t}, \quad S = (A + B)/2, \quad K = \sqrt{3}C/2.$$

The difference determines the stationary gradient at any measurement time. It must be unchanged between the two observations if the assumed model holds.

Step 2: Invert the two-time averages. Put $r = e^{-\pi^2 \Delta} \in (0, 1)$. From $m_0 = S + K$, $m_\Delta = S + rK$,

$$K = \frac{m_0 - m_\Delta}{1 - r}, \quad S = \frac{m_\Delta - rm_0}{1 - r}, \quad A = S - d/2, \quad B = S + d/2, \quad C = 2K/\sqrt{3}.$$

The denominator is nonzero because $\Delta > 0$. Equality of the observed differences is the only additional consistency condition; with it these parameters reconstruct all four readings under the stated model.

Step 3: Reconstruct the numerical example. Here $r = 1/2$, $d = 2$, $m_0 = 3$ and $m_\Delta = 5/2$. Thus $S = 2$, $K = 1$ and

$$A = 1, \quad B = 3, \quad C = 2/\sqrt{3}, \quad u = 1 + 2x + \frac{2}{\sqrt{3}}e^{-\pi^2 t} \sin(\pi x).$$

The endpoint values are one and three. The sensor transients are one at zero and one half at Δ , giving exactly the four supplied readings. The linear stationary part and the decaying sine verify $u_t = u_{xx}$.

Step 4: State the identifiability and conditioning limits. A single pair gives d and $S + K$, leaving infinitely many choices of S, K . For two times, errors e_0, e_Δ in the averages cause $\delta S = (e_\Delta - re_0)/(1 - r)$. If $|e_0|, |e_\Delta| \leq \eta$,

$$|\delta S| \leq \frac{1 + r}{1 - r} \eta \sim \frac{2\eta}{\pi^2 \Delta} \quad (\Delta \downarrow 0).$$

Opposite-signed errors attain the bound. Very close observations are therefore poorly conditioned even though the noiseless parameters are unique.