

Question 6

A semi-infinite rod has $u_t = \kappa u_{xx}$ on $x > 0$, with $\kappa, \omega > 0$ and a maintained oscillating boundary $u(0, t) = A + B \cos(\omega t)$, where $A > |B| > 0$. Seek its periodic response of the form

$$u(x, t) = A + Be^{-\alpha x} \cos(\omega t - \alpha x), \quad \alpha > 0,$$

with $u(x, t) \rightarrow A$ as $x \rightarrow \infty$. The initial field is to be the trace of this periodic response, so no additional startup transient is requested. Use the rightward flux convention $j = -\kappa u_x$.

Tasks

1. Determine α by direct substitution, and state the initial field that makes the response a solution of the full initial-boundary problem.
2. Derive the amplitude attenuation and phase delay at distance x . Find the depth where the oscillation amplitude falls to $1/e$ of its boundary value, and express it in terms of the period $P = 2\pi/\omega$.
3. Compute the boundary flux and its phase relative to the imposed temperature. Explain why constant far-field temperature does not imply zero instantaneous boundary flux.
4. Prove positivity of the temperature and approach to the far-field value as $x \rightarrow \infty$, uniformly in time. For $A = 2, B = 1, \kappa = 1, \omega = 2$, sketch profiles at phases $\omega t = 0, \pi/2, \pi$.

Question 6 – Solution

Strategy. A periodic boundary signal diffuses inward with both amplitude attenuation and a spatial phase shift.

Step 1: Match the PDE and the initial trace. Put $\theta = \omega t - \alpha x$. Then $u_t = -\omega B e^{-\alpha x} \sin \theta$ and $u_{xx} = -2\alpha^2 B e^{-\alpha x} \sin \theta$. Thus $\alpha = \sqrt{\omega/(2\kappa)}$. The boundary trace is the imposed cosine, and the compatible initial field is $f(x) = A + B e^{-\alpha x} \cos(\alpha x)$. It is generally not the constant A ; a constant initial field would require an additional transient.

Step 2: Quantify attenuation and delay. The local oscillation amplitude is $|B|e^{-\alpha x}$ and its phase lag is αx . Relative to the boundary waveform, the time delay is $\alpha x/\omega$, understood modulo the period. The penetration depth is

$$d = 1/\alpha = \sqrt{2\kappa/\omega} = \sqrt{\kappa P/\pi}.$$

Longer periods penetrate farther, with depth proportional to the square root of the period rather than the period itself.

Step 3: Determine the phase of the boundary flux. Differentiation gives $u_x = \alpha B e^{-\alpha x} (-\cos \theta + \sin \theta)$. Hence

$$j(0, t) = B\sqrt{\kappa\omega} \cos(\omega t + \pi/4).$$

The flux leads the boundary temperature oscillation by $\pi/4$ in phase and has zero average over a full period. Instantaneous storage and release near the driven boundary are compatible with a constant far-field limit.

Step 4: Verify bounds and interpret the profiles. For all $x \geq 0, t$, $|u - A| \leq |B|e^{-\alpha x} \leq |B|$, so $u \geq A - |B| > 0$. Also $\sup_t |u(x, t) - A| \leq |B|e^{-\alpha x} \rightarrow 0$ as $x \rightarrow \infty$. This is a spatial far-field limit uniform in time, not decay of the maintained oscillation at a fixed point as time increases. For the plotted parameters, $\alpha = 1$ and the penetration depth is one.

