

Question 5

Let both endpoint temperatures rise at a constant rate $r > 0$:

$$u_t = \kappa u_{xx}, \quad 0 < x < L, \quad u(0, t) = A + rt, \quad u(L, t) = B + rt,$$

where $\kappa, L > 0$ and $u(x, 0) = \ell(x) := A + (B - A)x/L$. You may use the sine expansion of $x(L - x)$.

Tasks

1. Subtract the instantaneous boundary line $\ell(x) + rt$ and derive the PDE and initial data for the remaining field. Explain why that line alone is not a solution.
2. Find a time-independent zero-endpoint correction $P(x)$ so that $\ell + rt + P$ solves the PDE. Add the transient needed to recover the original initial field.
3. Verify the reconstructed solution, justify the initial trace and positive-time differentiations, and determine its limiting offset relative to the moving boundary line.
4. Compute the limiting midpoint lag, the limiting mean lag and the eventual total-heat growth rate. Explain why there is no stationary equilibrium even though the offset approaches a fixed shape.

Question 5 – Solution

Strategy. A boundary lifting with nonzero time derivative creates a source in the transformed equation; a persistent lag balances that source.

Step 1: Keep the source created by the lifting. For $w = u - \ell - rt$,

$$w_t = \kappa w_{xx} - r, \quad w(0, t) = w(L, t) = 0, \quad w(x, 0) = 0.$$

The line $\ell + rt$ has time derivative r and zero second spatial derivative, so it fails the original source-free PDE when $r > 0$.

Step 2: Find the lag profile and the initial correction. A fixed P must satisfy $\kappa P'' = r$, $P(0) = P(L) = 0$, giving $P = -rx(L-x)/(2\kappa)$. The remaining transient has initial data $-P$. Its odd sine coefficients are $4rL^2/(\kappa\pi^3n^3)$. Thus

$$u = \ell + rt + P + \frac{4rL^2}{\kappa\pi^3} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t} \sin(n\pi x/L)}{n^3}.$$

Step 3: Verify all data and the moving-frame limit. The field $\ell + rt + P$ satisfies $u_t = \kappa u_{xx} = r$; its added transient satisfies the homogeneous heat equation. All correction terms vanish at the endpoints. At $t = 0$, the summable sine expansion equals $-P$ uniformly and recovers ℓ . For $t \geq \tau > 0$, the exponential transient permits all termwise derivatives. Initial corner smoothness is not implied: the boundary time derivative is r , whereas $\kappa \ell'' = 0$. As $t \rightarrow \infty$, the transient tends uniformly to zero, so $u - \ell - rt \rightarrow P$ uniformly.

Step 4: Quantify the persistent lag and heat input. The midpoint offset is $P(L/2) = -rL^2/(8\kappa)$, and the mean offset is $L^{-1} \int P = -rL^2/(12\kappa)$. Their negatives are the positive lags behind the boundary line. Since $P'(L) - P'(0) = rL/\kappa$, the eventual flux balance is $H' = \kappa[u_x]_0^L \rightarrow rL$, also obtained by differentiating the integrated solution. Boundary values grow indefinitely; it is the offset, not the absolute temperature, that approaches a fixed state.