

Question 4

On $0 < x < 1$, let the storage and conductivity vary so that

$$\frac{u_t}{1+x} = ((1+x)u_x)_x, \quad u(0,t) = 1, \quad u(1,t) = 3.$$

Define $j = -(1+x)u_x$, $\ell = \log 2$, and let the initial field be

$$f(x) = 1 + \frac{2 \log(1+x)}{\ell} + \sin\left(\frac{\pi \log(1+x)}{\ell}\right).$$

Tasks

1. Derive the stationary temperature and stationary flux. Explain why the straight line $1 + 2x$ is not stationary for this equation.
2. Use $s = \log(1+x)$ to transform the entire PDE and its endpoints. Construct and verify the full solution for the given initial field.
3. Compute the stored quantity $H(t) = \int_0^1 u(x,t)/(1+x) dx$ and verify $H' = j(0,t) - j(1,t)$.
4. For the error $v = u - S$, derive the weighted energy identity. Evaluate its weighted squared norm for this solution and explain why the weight cannot simply be omitted from the identity.

Question 4 – Solution

Strategy. The stationary gradient and the natural storage norm both follow from the variable coefficients.

Step 1: Integrate constant stationary flux. The stationary equation is $((1+x)S')' = 0$, so $(1+x)S' = C$. The endpoint difference gives $C = 2/\ell$, hence

$$S = 1 + 2 \log(1+x)/\ell, \quad j_S = -2/\ell.$$

For $1+2x$, the quantity $((1+x)u_x)' = 2$ is nonzero, so it is not stationary. Equal endpoint values do not make different interior profiles solve the same PDE.

Step 2: Transform and solve the full problem. Write $U(s, t) = u(e^s - 1, t)$ on $0 < s < \ell$. Then $u_x = U_s/(1+x)$ and $((1+x)u_x)_x = U_{ss}/(1+x)$. The transformed equation is $U_t = U_{ss}$, with endpoints one and three. Subtracting $1 + 2s/\ell$ gives one sine mode. With $\lambda = \pi^2/\ell^2$,

$$u = S + e^{-\lambda t} \sin(\pi \log(1+x)/\ell).$$

The chain-rule identities verify the original PDE, and the sine gives zero endpoint correction and exactly the prescribed initial field.

Step 3: Check storage against boundary flux. Since $dx/(1+x) = ds$,

$$H(t) = 2\ell + \frac{2\ell}{\pi} e^{-\lambda t}, \quad H' = -\frac{2\pi}{\ell} e^{-\lambda t}.$$

Also $j = -U_s = -2/\ell - (\pi/\ell)e^{-\lambda t} \cos(\pi s/\ell)$. Its left value minus its right value equals the displayed derivative. The unweighted integral of u is not the stored quantity in this model.

Step 4: Derive the weighted error dissipation. Multiplying $v_t/(1+x) = ((1+x)v_x)_x$ by v and integrating, with zero error at both ends, gives

$$\frac{1}{2} \frac{d}{dt} \int_0^1 \frac{v^2}{1+x} dx = - \int_0^1 (1+x) v_x^2 dx.$$

For this mode the squared norm is $(\ell/2)e^{-2\lambda t}$ and the right-hand integral is $(\pi^2/(2\ell))e^{-2\lambda t}$, agreeing with the identity. The weight is the storage coefficient; dropping it changes the energy calculation.