

Question 2

A rod on $0 < x < 1$ obeys $u_t = u_{xx}$ and is initially at zero temperature in $L^2(0, 1)$. At $t = 0$ the left endpoint is raised to one and held there, while the right remains zero: $u(0, t) = 1$, $u(1, t) = 0$ for $t > 0$. You may use Fourier sine convergence, Parseval, and $\int_0^\infty e^{-\pi^2 s^2} ds = 1/(2\sqrt{\pi})$.

Tasks

1. Subtract the stationary field, compute the sine coefficients of the transformed initial data and construct the full solution.
2. Justify the PDE for positive time and the L^2 initial trace. State the initial limit at interior points and at the left endpoint; explain why uniform convergence to the zero initial field is impossible.
3. Derive a convergent total-heat formula and prove that total heat increases for every positive time, although the source-free PDE has no interior forcing.
4. Find the rightward flux entering at $x = 0$ and its leading behavior as $t \downarrow 0$. Use a Riemann-sum limit to justify the singular rate rather than evaluating a divergent series at $t = 0$.

Question 2 – Solution

Strategy. The sudden boundary change creates an initial corner mismatch; subtracting the stationary profile isolates its decaying sine expansion.

Step 1: Construct the boundary step response. The stationary field is $S = 1 - x$. For $v = u - S$, the endpoints are zero and $v(x, 0) = x - 1$. Integration gives $2 \int_0^1 (x - 1) \sin(n\pi x) dx = -2/(n\pi)$. Hence

$$u(x, t) = 1 - x - \frac{2}{\pi} \sum_{n \geq 1} \frac{e^{-n^2 \pi^2 t} \sin(n\pi x)}{n}.$$

Step 2: Verify the appropriate initial trace. For $t \geq \tau > 0$, Gaussian decay makes every finitely differentiated transient series uniformly convergent. It satisfies the heat equation and zero endpoint values; adding S gives the required boundary data. The square-summable coefficients and Parseval prove $v \rightarrow x - 1$ in L^2 , hence $u \rightarrow 0$ in L^2 . At each interior point, the Fourier limit is zero after reconstruction; Gaussian damping preserves that limit by summation by parts. At $x = 0$, the positive-time value is always one, so its limit is one. Uniform convergence to zero on the closed interval is therefore impossible. No corner-continuous solution is claimed.

Step 3: Integrate the boundary-driven heating. For positive time,

$$H(t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n \text{ odd}} \frac{e^{-n^2 \pi^2 t}}{n^2}, \quad H'(t) = 4 \sum_{n \text{ odd}} e^{-n^2 \pi^2 t} > 0.$$

The initial norm trace gives $H(0+) = 0$ and the final heat is $1/2$. Differentiating the spatial series gives $H' = [u_x]_0^1 = j(0, t) - j(1, t)$, where $j = -u_x$. The endpoint supplies heat; no interior source is needed.

Step 4: Quantify the singular initial input. At the left endpoint,

$$j(0, t) = 1 + 2 \sum_{n \geq 1} e^{-n^2 \pi^2 t}.$$

With $h = \sqrt{t}$, the decreasing, integrable Gaussian gives $h \sum_{n \geq 1} e^{-\pi^2 (nh)^2} \rightarrow \int_0^\infty e^{-\pi^2 s^2} ds$ by upper/lower integral bounds. Thus $\sqrt{t} j(0, t) \rightarrow 1/\sqrt{\pi}$, or $j(0, t) \sim 1/\sqrt{\pi t}$. The singular rate is integrable in time and is consistent with vanishing initial heat. Substituting $t = 0$ into the differentiated series would be invalid.