

Question 10

Let $u_t = \kappa u_{xx}$ on $(0, L)$ with $\kappa, L > 0$, zero endpoint values, and initial field $f \in L^2(0, L)$. Let b_n be its sine coefficients and $E(t) = \frac{1}{2} \int_0^L u(x, t)^2 dx$ its quadratic energy. You may use Parseval's identity, positive-time termwise differentiation and sine completeness.

Tasks

1. Derive the energy identity and prove $E(t) \leq e^{-2\kappa(\pi/L)^2 t} E(0)$. Characterize every nonzero datum attaining equality at a fixed positive time.
2. Suppose $f(L - x) = -f(x)$ almost everywhere. Prove that only even modes remain and obtain the improved bound with rate $8\kappa\pi^2/L^2$. Identify when that bound is attained.
3. Show that a nonnegative, nonzero datum cannot have $b_1 = 0$. Explain why the enhanced asymptotic decay caused by removing the first mode requires sign changes in nonzero real initial data.
4. Prove the long-time L^2 asymptotic formula $e^{\kappa(\pi/L)^2 t} u \rightarrow b_1 \sin(\pi x/L)$, with a quantitative remainder bound. State what the formula says when $b_1 = 0$.

Question 10 – Solution

Strategy. Orthogonality turns energy decay and symmetry restrictions into precise statements about which coefficients are present.

Step 1: Prove the sharp basic energy decay. For $t > 0$, integration by parts gives $E' = \kappa \int u u_{xx} = -\kappa \int u_x^2$ because $u = 0$ at both ends. With $\lambda_1 = (\pi/L)^2$, Parseval yields

$$E(t) = \frac{L}{4} \sum_{n \geq 1} b_n^2 e^{-2\kappa n^2 \lambda_1 t} \leq e^{-2\kappa \lambda_1 t} E(0).$$

For fixed $t > 0$, every term with $n \geq 2$ has strictly smaller decay factor. Equality for nonzero data holds exactly when $f = b_1 \sin(\pi x/L)$ with $b_1 \neq 0$. This also proves sharpness of the uniform energy rate.

Step 2: Use antisymmetry to improve the rate. Changing variable $x \mapsto L - x$ in the coefficient integral and using $\sin(n\pi(L - x)/L) = (-1)^{n+1} \sin(n\pi x/L)$ gives $b_n = (-1)^n b_n$. Thus odd coefficients vanish. The smallest remaining index is two, giving

$$E(t) \leq e^{-8\kappa \pi^2 t/L^2} E(0).$$

Nonzero equality data are precisely scalar multiples of $\sin(2\pi x/L)$. The symmetry is preserved by the evolved even-mode series.

Step 3: Identify the sign constraint. If $f \geq 0$ almost everywhere and is nonzero in L^2 , then it is positive on a set of positive measure. The first sine is strictly positive on $(0, L)$; therefore $b_1 = (2/L) \int f \sin(\pi x/L) > 0$. Similarly a nonpositive nonzero datum has $b_1 < 0$. Thus any nonzero real datum with $b_1 = 0$ must have both positive and negative parts of positive measure. Removing the first mode is possible, but not for a nontrivial nonnegative initial temperature relative to the zero boundaries.

Step 4: Prove the normalized asymptotic statement. After removing the first term, Parseval gives

$$\|e^{\kappa \lambda_1 t} u - b_1 \sin(\pi x/L)\|_2^2 = \frac{L}{2} \sum_{n \geq 2} b_n^2 e^{-2\kappa(n^2-1)\lambda_1 t} \leq e^{-6\kappa \lambda_1 t} \|f\|_2^2.$$

The remainder norm is at most $e^{-3\kappa \lambda_1 t} \|f\|_2$ and tends to zero. If $b_1 = 0$, the normalized limit is zero and the actual field decays at least at the second-mode norm rate $e^{-4\kappa \lambda_1 t}$. The next nonzero coefficient, if present, can give an even faster decay rate.