

Question 9

For $u_t = u_{xx}$ on $(0, \pi)$ with zero endpoint values, let $\phi_n(x) = \sqrt{2/\pi} \sin(nx)$ be the orthonormal spatial modes. Write $u(x, 0) = \sum c_n \phi_n(x)$ and $u(x, T) = \sum d_n \phi_n(x)$, where $T > 0$. All norms in this question are $L^2(0, \pi)$ norms.

Tasks

1. Relate c_n and d_n . Prove that there is at most one L^2 initial field for given exact final data, and state the precise coefficient condition for such an initial field to exist.
2. Use initial fields ϕ_n to show that uniqueness of backward reconstruction does not imply continuous dependence on final data in these norms.
3. Suppose measurement noise in the final field has norm at most δ . Reconstruct only modes $1, \dots, N$ and prove that the reconstructed noise has norm at most $e^{N^2 T} \delta$. Show that this bound is sharp.
4. For $T = 0.1$, $\delta = 10^{-6}$, choose the largest integer N certified to keep reconstructed noise below 0.01. Plot $\log_{10}$ of modal amplification against integer mode number, and explain why controlling noise alone does not bound the omitted true initial modes.

Question 9 – Solution

Strategy. Each forward-decaying mode becomes an exponentially amplified mode when time is reversed.

Step 1: Characterize exact inversion. Forward evolution gives $d_n = c_n e^{-n^2 T}$, so the only possible coefficients are $c_n = e^{n^2 T} d_n$. Completeness implies at most one initial field. It exists in L^2 exactly when

$$\sum_{n \geq 1} e^{2n^2 T} |d_n|^2 < \infty.$$

Indeed this condition constructs an L^2 sine expansion whose forward solution has the given coefficients. Arbitrary L^2 final data need not satisfy this stronger condition.

Step 2: Disprove continuity despite uniqueness. Take $f_n = \phi_n$, whose initial norm is one. Its final field has norm $e^{-n^2 T} \rightarrow 0$. Thus final data tend to zero while their uniquely determined initial fields do not. The backward map on its attainable-data domain is not continuous at zero in the stated norms. No contradiction arises with the stable forward problem, which damps these same modes.

Step 3: Bound noise after truncating the inverse. Let η_n be the noise coefficients, so $\sum |\eta_n|^2 \leq \delta^2$. The retained reconstructed noise satisfies

$$\left\| \sum_{n=1}^N e^{n^2 T} \eta_n \phi_n \right\|_2^2 = \sum_{n=1}^N e^{2n^2 T} |\eta_n|^2 \leq e^{2N^2 T} \delta^2.$$

Taking square roots gives the bound, attained by noise $\delta \phi_N$.

Step 4: Choose a noise cutoff and state its limitation. The requirement is $e^{0.1 N^2} 10^{-6} < 10^{-2}$, or $N^2 < 40 \log 10$. Thus $\lfloor N = 9 \rfloor$ is the largest certified integer: the bounds for 9, 10 are approximately 0.003295 and 0.022027. The plot uses discrete stems, since n is an integer; their heights are $0.1 n^2 / \log 10$. The omitted initial tail has norm $(\sum_{n > N} |c_n|^2)^{1/2}$, for which no numerical bound was supplied. A noise cutoff controls amplification but requires additional prior information to control that approximation error.

