

Question 8

For $\alpha > 0$, solve the forced heat problem

$$u_t = u_{xx} + e^{-\alpha t} \sin x, \quad 0 < x < \pi, \quad t > 0, \quad u(0, t) = u(\pi, t) = 0, \quad u(x, 0) = 0.$$

Use a field of the form $u(x, t) = q_\alpha(t) \sin x$. This is interior forcing; the boundary temperatures remain zero.

Tasks

1. Derive and solve the scalar initial-value problem for q_α , treating $\alpha = 1$ separately.
2. Verify the PDE and all data, and prove that the constructed amplitude is positive for $t > 0$. Explain why uniqueness holds in the class of smooth solutions continuous into L^2 initially.
3. Determine the unique time of maximum amplitude for $\alpha \neq 1$ and its limit as $\alpha \rightarrow 1$. Check the maximum directly for $\alpha = 1$.
4. Derive the total-heat balance and compare matching temporal decay rates here with unbounded oscillatory resonance. Does the factor t in the exceptional solution cause unbounded temperature?

Question 8 – Solution

Strategy. The source has one spatial eigenfunction, so the full problem reduces to a forced scalar decay equation.

Step 1: Solve the amplitude equation including the exceptional rate. Substitution gives $q'_\alpha + q_\alpha = e^{-\alpha t}$, $q_\alpha(0) = 0$. Multiplying by e^t and integrating yields

$$q_\alpha(t) = \begin{cases} (e^{-\alpha t} - e^{-t})/(1 - \alpha), & \alpha \neq 1, \\ te^{-t}, & \alpha = 1. \end{cases}$$

The exceptional formula follows from the same integrating-factor integral; it is not obtained by substituting into a zero denominator.

Step 2: Verify existence, sign and uniqueness. The scalar ODE verifies the PDE, while the sine factor and $q_\alpha(0) = 0$ verify endpoints and initial data. The integral representation

$$q_\alpha(t) = \int_0^t e^{-(t-s)} e^{-\alpha s} ds$$

is strictly positive for $t > 0$. The difference of two solutions with the same source solves the homogeneous zero-data problem. Its squared L^2 norm is nonincreasing by integration by parts; starting at a positive time and using the zero initial norm proves uniqueness in the stated class.

Step 3: Locate the unique maximum. For $\alpha \neq 1$, $q'_\alpha = (e^{-t} - \alpha e^{-\alpha t})/(1 - \alpha)$. The equation $q'_\alpha = 0$ is $e^{(\alpha-1)t} = \alpha$, giving

$$t_\alpha = \frac{\log \alpha}{\alpha - 1} > 0.$$

There is exactly one root; $q'_\alpha(0) = 1 > 0$ and the derivative is negative for sufficiently large time, proving it is the unique maximum. As $\alpha \rightarrow 1$, $t_\alpha \rightarrow 1$. Directly $(te^{-t})' = e^{-t}(1 - t)$, with maximum $1/e$ at $t = 1$.

Step 4: Interpret the forced heat balance. The total heat is $H = 2q_\alpha$. Since $[u_x]_0^\pi = -2q_\alpha$ and the source integral is $2e^{-\alpha t}$,

$$H' = -2q_\alpha + 2e^{-\alpha t},$$

which agrees with the scalar ODE. At $\alpha = 1$, matching the source decay rate to the homogeneous decay produces te^{-t} . This field is bounded and tends to zero; its time factor does not imply unbounded oscillatory resonance. For every fixed $\alpha > 0$, the source and the solution ultimately decay.