

Question 7

For the zero-endpoint heat problem $u_t = u_{xx}$ on $(0, \pi)$, let $f \in L^2(0, \pi)$ with $\|f\|_2 \leq M$, where $M > 0$. Its solution for $t > 0$ is

$$u = \sum_{n \geq 1} b_n e^{-n^2 t} \sin(nx), \quad u_N = \sum_{n=1}^N b_n e^{-n^2 t} \sin(nx), \quad N \geq 1.$$

You may use Parseval: $\sum b_n^2 = 2\|f\|_2^2/\pi$, and Cauchy-Schwarz.

Tasks

1. Prove $\|u - u_N\|_2 \leq M e^{-(N+1)^2 t}$. Show that the bound is sharp over the stated data class for each N, t .
2. Derive a uniform-in- x error bound using Cauchy-Schwarz and the tail sum $\sum_{n > N} e^{-2n^2 t}$.
3. Bound that sum by a geometric series using $n = N + 1 + j$, $j \geq 0$, and obtain an explicit bound requiring no infinite summation.
4. For $M = 1, t \geq 0.1$, certify that six modes give a uniform error below 0.01. Explain why the available bound does not certify five modes, and why this does not prove that five modes fail for every particular initial field.

Question 7 – Solution

Strategy. A truncation certificate needs a bound on the omitted coefficients, not just a graph of the retained modes.

Step 1: Prove the sharp norm estimate. Orthogonality gives

$$\|u - u_N\|_2^2 = \frac{\pi}{2} \sum_{n>N} b_n^2 e^{-2n^2 t} \leq e^{-2(N+1)^2 t} \|f\|_2^2.$$

Taking square roots proves the bound. Equality is attained by $f = M\sqrt{2/\pi} \sin((N+1)x)$: all its energy lies in the first omitted mode. Thus the rate and constant are sharp for the whole L^2 ball.

Step 2: Control pointwise errors uniformly. Using $|\sin nx| \leq 1$ and Cauchy–Schwarz,

$$|u - u_N| \leq \left(\sum_{n>N} b_n^2 \right)^{1/2} \left(\sum_{n>N} e^{-2n^2 t} \right)^{1/2} \leq \sqrt{\frac{2}{\pi}} M \left(\sum_{n>N} e^{-2n^2 t} \right)^{1/2}.$$

This bound is independent of x and finite for $t > 0$. It also justifies uniform convergence of the positive-time solution for arbitrary L^2 data.

Step 3: Replace the tail by a geometric bound. For the integer $j \geq 0$, $(N+1+j)^2 \geq (N+1)^2 + (2N+3)j$, since $j^2 \geq j$. Therefore

$$\sum_{n>N} e^{-2n^2 t} \leq \frac{e^{-2(N+1)^2 t}}{1 - e^{-2(2N+3)t}}, \quad \|u - u_N\|_\infty \leq \sqrt{\frac{2}{\pi}} \frac{M e^{-(N+1)^2 t}}{\sqrt{1 - e^{-2(2N+3)t}}}.$$

The denominator is positive because $t > 0$. Each factor makes this certificate decrease as time increases.

Step 4: Make a numerical guarantee with the correct scope. At $t = 0.1$, the bound is less than 0.00611 for $N = 6$ and greater than 0.0226 for $N = 5$. Hence six modes certify the requested 0.01 uniform error for every $t \geq 0.1$ and every datum with $\|f\|_2 \leq 1$. The five-mode certificate is too large; it does not prove failure for each field. For example, a datum supported entirely in the first five modes has zero truncation error with five modes. The bound is a sufficient test, not an exact error formula for unknown data.