

Question 5

Let $\kappa, L > 0$. Solve $u_t = \kappa u_{xx}$ on $0 < x < L$, $t > 0$, with insulated ends $u_x(0, t) = u_x(L, t) = 0$ and initial temperature $f(x) = x$. Use the convention $f = a_0 + \sum_{n \geq 1} a_n \cos(n\pi x/L)$, where $a_0 = L^{-1} \int_0^L f$ and $a_n = (2/L) \int_0^L f \cos(n\pi x/L)$. You may use cosine-series convergence and Parseval's identity.

Tasks

1. Compute the coefficients and construct the solution, retaining the zero mode.
2. Prove conservation of the mean and the reflection identity $u(L - x, t) = L - u(x, t)$. Determine the limiting equilibrium.
3. Justify uniform recovery of the initial values, and explain why the spatial derivative cannot extend continuously through the initial boundary corners.
4. Compute $V(t) = \int_0^L (u - L/2)^2 dx$ as a series, find $V(0)$ and establish $V(t) \leq e^{-2\kappa(\pi/L)^2 t} V(0)$. Sketch profiles for $L = \kappa = 1$ at $t = 0, 0.03, 0.15$.

Question 5 – Solution

Strategy. Insulation preserves the constant component while all nonconstant cosine modes decay.

Step 1: Compute and evolve the cosine coefficients. Integration gives $a_0 = L/2$ and $a_n = 2L((-1)^n - 1)/(\pi^2 n^2)$. Therefore

$$u(x, t) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t} \cos(n\pi x/L)}{n^2}.$$

Gaussian decay justifies the PDE and derivative boundary conditions termwise for $t > 0$. The constant mode remains stationary.

Step 2: Identify the conserved and symmetric parts. Integrating the PDE yields $(\int u)' = \kappa[u_x]_0^L = 0$, so the mean is $L/2$. For odd n , $\cos(n\pi(L-x)/L) = -\cos(n\pi x/L)$. The series therefore gives $u(L-x, t) = L - u(x, t)$, in particular $u(L/2, t) = L/2$ at all times. The absolutely summable coefficients and positive modal decay imply uniform convergence to the equilibrium $L/2$ as $t \rightarrow \infty$.

Step 3: Check values and derivatives separately. The n^{-2} coefficients give absolute uniform convergence at $t = 0$; the cosine convergence theorem identifies the sum as x , including both endpoints. Summable domination proves uniform recovery of those values as $t \downarrow 0$. But the initial trace has derivative one, whereas $u_x = 0$ at the endpoints for every positive time. A continuously extending spatial derivative would have to take both values at each initial corner, which is impossible. Uniform convergence of temperatures does not imply convergence of derivatives.

Step 4: Quantify equilibration. Orthogonality gives

$$V(t) = \frac{8L^3}{\pi^4} \sum_{n \text{ odd}} \frac{e^{-2\kappa(n\pi/L)^2 t}}{n^4}, \quad V(0) = \int_0^L (x - L/2)^2 dx = \frac{L^3}{12}.$$

Each exponential is at most $e^{-2\kappa(\pi/L)^2 t}$, proving the required bound. The profiles flatten toward their conserved mean, rather than toward zero. Only the initial profile has the incompatible endpoint slopes.

