

Question 4

Solve $u_t = u_{xx}$ on $0 < x < \pi$, $t > 0$, with $u(0, t) = u(\pi, t) = 0$ and initial data in the L^2 sense

$$f(x) = \begin{cases} 1, & 0 < x < \pi/2, \\ 1/2, & x = \pi/2, \\ 0, & \pi/2 < x < \pi. \end{cases}$$

The L^2 norm is $\|v\|_2^2 = \int_0^\pi |v|^2 dx$. You may use sine-series convergence at continuity points and to the average at a jump, and Parseval's identity.

Tasks

1. Compute the sine coefficients and construct the heat solution.
2. Prove that the solution is smooth and solves the PDE for positive time, and that it approaches f in L^2 as $t \downarrow 0$.
3. State the pointwise limit at the interior jump and at each boundary when boundary values are retained as zero. Explain why convergence cannot be uniform on $[0, \pi]$, even if the endpoints of f are defined to be zero.
4. Prove uniqueness among solutions smooth up to the spatial boundaries for $t > 0$ and continuous into L^2 at $t = 0$, with these zero endpoint values. Use an energy estimate starting at $t = \delta > 0$ and then let $\delta \downarrow 0$.

Question 4 – Solution

Strategy. Interpret the initial condition in its stated norm; incompatible corners and jumps do not invalidate a positive-time series solution.

Step 1: Compute all modal amplitudes. Integration over the nonzero half of the interval gives

$$b_n = \frac{2}{\pi n} (1 - \cos(n\pi/2)), \quad u(x, t) = \sum_{n \geq 1} b_n e^{-n^2 t} \sin(nx).$$

The point value assigned at $\pi/2$ does not affect the integral. Modes divisible by four vanish; the other coefficients must not all be replaced by the odd-only coefficients for a constant temperature on the entire interval.

Step 2: Verify positive-time smoothness and the norm trace. For $t \geq \tau > 0$, $|b_n| \leq 4/(\pi n)$ and the exponential makes all finitely differentiated series uniformly convergent. Thus the PDE and endpoints hold. Parseval and the sine expansion of f give

$$\|u(\cdot, t) - f\|_2^2 = \frac{\pi}{2} \sum_{n \geq 1} b_n^2 (1 - e^{-n^2 t})^2 \rightarrow 0,$$

by summable domination, since $\sum b_n^2 < \infty$. This is precisely the specified initial condition.

Step 3: State the pointwise limits correctly. The limits are 1 on $(0, \pi/2)$, 0 on $(\pi/2, \pi)$ and $1/2$ at the jump. To justify Gaussian damping of a pointwise convergent Fourier series, apply summation by parts: the decreasing weights $e^{-n^2 t}$ average its partial sums, have total weight tending to one, and move past each fixed index as $t \downarrow 0$. Thus they preserve the Fourier limit. At both endpoints u is identically zero for $t > 0$, so the limit is zero. A uniform limit of continuous functions on $[0, \pi]$ would be continuous; the jump and the left corner mismatch preclude uniform convergence to these data.

Step 4: Prove uniqueness in the stated class. Let w be the difference of two such solutions. For $t \geq \delta > 0$,

$$\frac{d}{dt} \|w\|_2^2 = 2 \int_0^\pi w w_{xx} dx = -2 \int_0^\pi w_x^2 dx \leq 0.$$

The boundary term vanishes. Hence $\|w(\cdot, t)\|_2 \leq \|w(\cdot, \delta)\|_2$. Both solutions have the same L^2 initial trace, so the right side tends to zero as $\delta \downarrow 0$. Positive-time continuity then makes $w = 0$ pointwise. No smoothness at the discontinuous initial trace was assumed.