

Question 3

For $u_t = \kappa u_{xx}$ on $0 < x < L$ with $\kappa, L > 0$, zero endpoint values, and $0 < a < L$, take the continuous tent-shaped initial temperature

$$f(x) = \begin{cases} x/a, & 0 \leq x \leq a, \\ (L-x)/(L-a), & a \leq x \leq L. \end{cases}$$

Its height is one. You may use Fourier convergence for continuous piecewise smooth functions with zero endpoint values.

Tasks

1. Derive all sine coefficients by splitting the integral at a , retaining the change in slope, and write the heat solution.
2. Justify the initial trace and positive-time smoothing even though f' jumps at a .
3. Show how the location a can be recovered from the ratio b_2/b_1 . State its allowed range and explain why the recovery is unique when $0 < a < L$.
4. For $L = \kappa = 1$, $a = 1/3$, simplify the coefficients, identify the missing modes, and sketch the initial tent and the profiles at $t = 0.01, 0.05, 0.15$.

Question 3 – Solution

Strategy. The corner in the initial profile appears as a slope jump in the coefficient calculation, not as a failure of the positive-time heat solution.

Step 1: Compute the contribution of the slope jump. With $k = n\pi/L$, integration by parts once gives $\int f \sin(kx) = k^{-1} \int f' \cos(kx)$ since f vanishes at both endpoints. Using $f' = 1/a$ to the left and $-1/(L-a)$ to the right gives

$$\int_0^L f \sin(kx) dx = \frac{L \sin(ka)}{a(L-a)k^2}.$$

Therefore

$$b_n = \frac{2L^2 \sin(n\pi a/L)}{\pi^2 n^2 a(L-a)}, \quad u = \sum_{n=1}^{\infty} b_n e^{-\kappa(n\pi/L)^2 t} \sin(n\pi x/L).$$

Step 2: Separate initial regularity from later regularity. Since $|b_n| \leq C/n^2$, the sine series converges absolutely and uniformly, and Fourier convergence identifies its initial sum as the continuous tent. Summable domination gives uniform convergence to f as $t \downarrow 0$. For every $\tau > 0$, polynomial factors from any finite number of derivatives are dominated by Gaussian decay for $t \geq \tau$. Thus the series is smooth for positive time, satisfies the PDE termwise and retains zero endpoints. Its initial derivative jump does not persist as a corner in a positive-time profile.

Step 3: Recover the peak location from two modes. Put $\theta = \pi a/L \in (0, \pi)$. Then $b_1 > 0$ and

$$\frac{b_2}{b_1} = \frac{\sin 2\theta}{4 \sin \theta} = \frac{1}{2} \cos \theta, \quad a = \frac{L}{\pi} \arccos(2b_2/b_1).$$

The ratio must lie strictly between $-1/2$ and $1/2$. Cosine is strictly decreasing on $(0, \pi)$, so this reconstruction is unique in the stated family.

Step 4: Specialize and interpret the profiles. For $L = \kappa = 1$, $a = 1/3$, $b_n = 9 \sin(n\pi/3)/(\pi^2 n^2)$. Exactly the multiples of three vanish, and $b_2/b_1 = 1/4$ recovers $a = 1/3$. The figure uses the exact piecewise initial trace and rapidly convergent positive-time series. The peak rounds as the profile diffuses toward zero.

