

Question 2

Let $\kappa, L > 0$ and solve $u_t = \kappa u_{xx}$ on $0 < x < L$, $t > 0$, with zero endpoint values and initial temperature $f(x) = x(L - x)$. You may use Fourier sine convergence for piecewise smooth odd extensions.

Tasks

1. Compute every coefficient $b_n = (2/L) \int_0^L f(x) \sin(n\pi x/L) dx$ and construct the solution series.
2. Justify uniform convergence to the initial trace as $t \downarrow 0$ and termwise verification of the PDE for $t > 0$. Do not infer corner smoothness from uniform convergence of the values.
3. Explain the absence of the even modes from reflection symmetry. Determine the total heat $H(t)$ as a convergent series and evaluate $H(0)$ directly.
4. Differentiate H for positive time and verify $H' = \kappa[u_x]_0^L$. Explain why H strictly decreases and why the solution is not a single separated product.

Question 2 – Solution

Strategy. Exact coefficients control the initial convergence, while Gaussian mode decay justifies later differentiation.

Step 1: Compute the sine coefficients. Put $k = n\pi/L$. Twice integrating by parts, using $f(0) = f(L) = 0$ and $f'' = -2$, gives $\int_0^L f \sin(kx) dx = 2(1 - (-1)^n)/k^3$. Thus $b_n = 4L^2(1 - (-1)^n)/(n^3\pi^3)$, and

$$u(x, t) = \frac{8L^2}{\pi^3} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{\sin(n\pi x/L)}{n^3} e^{-\kappa(n\pi/L)^2 t}.$$

Step 2: Verify the full initial-value solution. The coefficients are absolutely summable, so the series at $t = 0$ converges uniformly. Fourier convergence identifies its sum as f on $[0, L]$, including the zero endpoints. Dominated convergence of the coefficient sum then gives $\|u(\cdot, t) - f\|_\infty \rightarrow 0$. For $t \geq \tau > 0$, any fixed number of spatial or time derivatives only adds powers of n ; the factor $e^{-\kappa(n\pi/L)^2 \tau}$ makes these series uniformly summable. Hence $u_t = \kappa u_{xx}$ termwise for positive time. At an initial corner, $f'' = -2$ conflicts with the zero boundary time derivative; continuity of all second spatial derivatives through those corners is not claimed.

Step 3: Use symmetry and integrate the solution. Since $f(L - x) = f(x)$ and $\sin(n\pi(L - x)/L) = (-1)^{n+1} \sin(n\pi x/L)$, even coefficients vanish. Integrating the uniformly convergent series gives

$$H(t) = \frac{16L^3}{\pi^4} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t}}{n^4}, \quad H(0) = \int_0^L x(L - x) dx = \frac{L^3}{6}.$$

Step 4: Check heat loss and the mode content. For $t > 0$, differentiation gives

$$H'(t) = -\frac{16\kappa L}{\pi^2} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t}}{n^2} = \kappa[u_x]_0^L.$$

The equality on the right follows by differentiating the spatial sine series and using the odd-mode cosine endpoint values. Every summand is positive, so $H' < 0$. Infinitely many nonzero coefficients decay at distinct rates; in particular the ratio of the third coefficient to the first changes with time. A single product would have fixed ratios wherever its time factor is nonzero.