

Question 10

For $u_t = u_{xx} + ru$ on $0 < x < \pi$, $t \geq 0$, impose homogeneous Dirichlet endpoints. The separated spatial modes are $\sin(nx)$ with eigenvalues n^2 , $n = 1, 2, \dots$. Consider $1 < r < 4$ and the finite-mode initial field

$$u(x, 0) = \sin(2x) + \varepsilon \sin x, \quad 0 < \varepsilon < 1.$$

Only this finite-mode construction is requested; no arbitrary-data expansion or infinite-series convergence theorem is needed.

Tasks

1. Derive the time equation for each mode and construct a solution with the given initial field. Verify its PDE and endpoints.
2. Compare the behavior when $\varepsilon = 0$ with every fixed $\varepsilon > 0$. Prove that in the latter case $e^{-(r-1)t}u$ converges uniformly to $\varepsilon \sin x$.
3. Using $\|v\|_2^2 = \int_0^\pi v^2 dx$, compute the squared norm of each modal component and determine the time when the growing component overtakes the decaying component in this norm.
4. For $r = 2$, $\varepsilon = 1/8$, sketch the two positive modal amplitudes and mark their crossing. Explain why finding a decaying separated solution alone does not show that all solutions decay, and state the first-mode threshold in r .

Question 10 – Solution

Strategy. Separation reveals individual growth rates; superposition shows how a small unstable component can dominate a decaying one.

Step 1: Construct and verify the finite sum. For $X_n = \sin nx$, $X_n'' = -n^2 X_n$, so the factor equation is $T_n' = (r - n^2)T_n$. Consequently

$$u = e^{(r-4)t} \sin 2x + \varepsilon e^{(r-1)t} \sin x.$$

Each time derivative has coefficient $r - n^2$, matching $u_{xx} + ru$. Both sine factors vanish at $0, \pi$, and their initial coefficients are $1, \varepsilon$. Linearity justifies adding these two verified modes.

Step 2: Compare exact cancellation with small contamination. For $\varepsilon = 0$, the solution decays uniformly because $r - 4 < 0$. For fixed $\varepsilon > 0$, the first mode grows because $r - 1 > 0$. Moreover

$$e^{-(r-1)t} u = \varepsilon \sin x + e^{-3t} \sin 2x, \quad \|e^{-(r-1)t} u - \varepsilon \sin x\|_\infty = e^{-3t}.$$

This proves uniform convergence of the normalized shape. The unnormalized solution grows without bound in sup norm, as is also seen at $x = \pi/2$.

Step 3: Compare modal norms exactly. Since $\int_0^\pi \sin^2(nx) dx = \pi/2$, the squared component norms are

$$N_2^2 = \frac{\pi}{2} e^{2(r-4)t}, \quad N_1^2 = \frac{\pi}{2} \varepsilon^2 e^{2(r-1)t}.$$

Their equality is $\varepsilon^2 e^{6t} = 1$, giving $t_c = \frac{1}{3} \log(1/\varepsilon)$. The growing component is larger for $t > t_c$. The crossing time is independent of r in the stated range because the difference of modal rates is always three. Orthogonality also gives $\|u\|_2^2 = N_1^2 + N_2^2$.

Step 4: Interpret the crossing and the threshold. For $r = 2$, $\varepsilon = 1/8$, the amplitudes are $a_1 = e^t/8$ and $a_2 = e^{-2t}$. They cross at $t_c = \log 2$, with common value $1/4$. A decaying mode only describes its own initial shape. A nonzero first-mode component decays for $r < 1$, is stationary for $r = 1$, and grows for $r > 1$. Thus one decaying example cannot establish decay for every admissible field.

